Optimal Liquidation Problems and HJB Equations with Singular Terminal Condition

DISSERTATION

zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)

im Fach
Mathematik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät
der Humboldt-Universität zu Berlin

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Tag der Disputation: 20. April 2017

Abstract

We study stochastic optimal control problems arising in the framework of optimal portfolio liquidation under limited liquidity. Our framework is flexible enough to allow for Markovian and non-Markovian impact functions and for simultaneous trading in primary venues and dark pools.

The key characteristic of portfolio liquidation models is the singular terminal condition of the value function that is induced by the liquidation constraint. For linear-quadratic models, the standard ansatz reduces the HJB equation for the value to a (system of) partial differential equation(s), backward stochastic differential equation(s) or backward stochastic partial differential equation(s) with singular terminal condition, depending on the choice of the cost coefficients.

We establish novel existence, uniqueness and regularity results for (BS)PDEs with singular terminal conditions arising in models of optimal portfolio liquidation, prove that the respective value functions can indeed be described by a (BS)PDE, and give the optimal trading strategies in feedback form.

For Markovian and non-Markovian impact models we establish a novel approach based on the precise asymptotics of the value function at the terminal time. For purely Markovian liquidation problems this allows us to establish the existence smooth solutions to singular PDEs. For a class mixed Markovian/non-Markovian models we characterize the HJB equation in terms of a singular BSPDE for which we establish existence and uniqueness of a solution using a stochastic penalization method.

Zusammenfassung

Gegenstand dieser Arbeit sind stochastische Kontrollprobleme im Kontext von optimaler Portfolioliquidierung in illiquiden Märkten. Dabei betrachten wir sowohl Markovsche sowie nicht-Markovsche Preiseinflussfunktionale und berücksichtigen den Handel sowohl im Primärmarkt als auch in Dark Pools.

Besonderes Merkmal von Liquidierungsproblemen ist die durch die Liquidierungsbedingung induzierte singuläre Endbedingung an die Wertfunktion. Der Standardansatz für linear-quadratische Probleme reduziert die HJB-Gleichungen für die Wertfunktion – je nach Zustandsdynamik – auf (ein System) partielle(r) Differentialgleichungen, stochastische(r) Rückwärtsdifferentialgleichungen beziehungsweise stochastische(r) partielle(r) Rückwärtsdifferentialgleichungen (BSPDE).

Wir beweisen neue Existenz-, Eindeutigkeits- und Regularitätsresultate für diese zur Lösung optimaler Liquidierungsprobleme verwendeten Differentialgleichungen mit singulärer Endbedingung, verifizieren die Charakterisierung der zugehörigen Wertfunktion anhand dieser Differentalgleichungen und geben die optimale Handelsstrategie in Feedbackform.

Für Markovsche und nicht-Markovsche Preiseinflussmodelle wird eine neuartiger Ansatz basierend auf der genauen singulären Asymptotik der Wertfunktion vorgelegt. Für vollständig Markovsche Liquidierungsprobleme erlaubt uns dieser, die Existenz glatter Lösungen der singulären partiellen Differentialgleichungen zu zeigen. Für eine Klasse von Problemen mit Markovscher/nicht-Markovscher Struktur charakterisieren wir die HJB-Gleichungen durch eine singuläre BSPDE, für die wir die Existenz und Eindeutigkeit einer Lösung über einen Bestrafungsansatz herleiten.

Acknowledgment

Many people have participated, directly or indirectly throughout my PhD work and I would like to name and thank a few of them for their valuable contributions.

First and foremost, I am deeply grateful to my advisor Prof. Dr. Ulrich Horst. Without his support this thesis would clearly not have been possible. He always was a great mentor by introducing me into financial mathematics and by his open-minded and critical advices. In spite of his many other duties, he always took the time for long and valuable discussions, providing a fruitful basis to develop my own ideas. I thank him for his persistent patience and his constant encouragement during the last years and last but not least for his excellent career advices.

Second, I would like to express my gratitude to Prof. Dr. Éric Séré and Dr. Jinniao Qiu for the very rewarding collaboration. It has been an honor and a privilege to work with them.

Moreover, I would like to thank Prof. Dr. Stefan Ankirchner, Dr. Julio Backhoff, Dr. Gonçalo dos Reis, Oliver Janke, Dr. Peter Kratz, Dr. Dörte Kreher and Dr. Thomas Kruse for the valuable discussions and comments.

Furthermore, I would like to thank Prof. Dr. Ronnie Sircar and Prof. Dr. H. Mete Soner for readily accepting to co-examine this thesis.

Last but not least, I gratefully acknowledge the financial support by *d-fine GmbH* through the PhD fellowship "Optimization in Financial Markets". In persona, my sincere thanks go to Dr. Jörn Rank for his extraordinary commitment to support academia and for the pleasant exchange at various events.

Contents

1. Introduction

Traditional financial market models assume that asset prices follow an exogenous stochastic process and that all transactions can be settled without any impact on market prices. This assumption is appropriate for small investors who trade only a negligible proportion of the average daily trading volume. It is not always appropriate, though, for institutional investors trading large blocks of shares over a short time span. Due to limited liquidity, large orders generate significant impact on the asset price, typically moving prices in an unfavorable direction.

In this thesis we consider stochastic control problems in continuous time arising in models of *optimal portfolio liquidation* (or *optimal order execution*, *optimal position closure*) under limited liquidity. In illiquid markets, trading typically has an impact on asset prices, hence generates liquidity costs. The difference between the realized price and the price before the trade is called *price impact* (or *market impact*). The financial mathematics literature distinguishes different forms of price impact, see Gökay et al. [GRS11] for an overview. The *instantaneous* (or *temporary*) impact only affects the current trade without any effects on future trades. The *permanent* impact affects current and future trades equally. Price impact is referred to as *persistent* (or *transient*) if the impact of current trades on future prices decays gradually over time.

The *execution costs* (or *implementation shortfall*) are the difference between the revenues if the whole position could be closed at given benchmark prices (*book value*) and the realized revenues from liquidating the position in a market with price impact. To minimize execution costs traders typically split their positions into smaller once which are then placed successively into the market. Splitting orders over time, however, entails *market risk*: slower trading increases the volatility of the portfolio value. Finding an optimal trading schedule is referred to as the optimal portfolio liquidation problem in the financial mathematics literature.

Optimal liquidation problems have received considerable attention in the mathematical finance and control literature in recent years. Bertsimas and Lo [BL98] were the first to consider the problem of minimizing expected execution costs. They solved the optimal liquidation problem for a risk-neutral investor in a discrete time model with linear instantaneous and linear permanent price impact. Almgren and Chriss [AC01] extended this model to risk-averse investors by considering a mean-variance optimization of the execution costs and give closed form solutions in continuous time. In Almgren [Alm03] the problem is formulated for general power-law (instantaneous and permanent) price impact functions; an empirical calibration suggesting a root-shaped instantaneous and a linear permanent impact can be found in Almgren et al. [ATHL05]. Huberman and Stanzl [HS04] show that only linear permanent price impact rules out arbitrage from price-manipulation strategies. Execution costs due to linear permanent impact, however, are independent of the particular trading schedule, and can hence be disregard when computing optimal liquidation strategies.

One of the main characteristics of stochastic control problems arising in the framework of optimal portfolio liquidation originates from the liquidation constraint: at the end of the liquidation period the open position has to be closed. In models with instantaneous price impact this *terminal state constraint* induces a singularity of the value function at the terminal time. When solving the stochastic control problem by means of Bellman's dynamic programming principle, relating the value function to the Hamilton-Jacobi-Bellman (HJB) equation, this induces a singular terminal condition to the HJB equation which renders the HJB characterization of the value function challenging from a mathematical perspective.

The singularity is already present, yet not immediately visible, in the original liquidation problem of Almgren and Chriss [AC01]. Within their mean variance framework and with *arithmetic* Brownian motion as the benchmark price process, the objective functional is deterministic, and the optimization problem is essentially a classical variational problem where the terminal state constraint causes no further difficulties. Schied et al. [SST10] extended the arithmetic Brownian framework of [AC01] to expected utility maximizing investors with constant absolute risk aversion and general super-linear instantaneous price impact. They show that optimal trading strategies are deterministic, which allows them to also use calculus of variation techniques rather than dynamic programming. However, when considering a *geometric* Brownian motion as the underlying price process as in Forsyth et al. [FKTW12], the optimal execution strategies become price-sensitive. One is then faced with a genuine stochastic control problem where the singularity becomes a challenge when determining the value function and applying verification arguments.

Several approaches to overcome this challenge have recently been suggested in the stochastic control literature. Gatheral and Schied [GS11] and Schied [Sch13b] consider a price-sensitive liquidation problem of fully linear-quadratic structure in both states, position and price. In this case the HJB reduces to a system of uncoupled ordinary differential equations (ODEs) with a singular terminal condition for which they obtain closed-form solutions. Closed-form solutions are also obtained in Kratz [Kra14], where an optimal liquidation problem with active and passive orders is considered that is not fully linear-quadratic but still allows by a quasi-polynomial ansatz to reduce the HJB equation to a system of uncoupled ODEs.

When more general cost functionals are considered, still being quadratic in the position but with general stochastic coefficients (Markovian, non-Markovian, or a mixture of both), then the standard quadratic ansatz for the value function (in the position) reduces the HJB equation—depending on the dynamics of the coefficients—to a partial differential equation (PDE), backward stochastic differential equation (BSDE), or backward stochastic partial differential equation (BSPDE) with singular terminal condition. In such a general framework closed-form solutions are typically not available and the analysis of portfolio liquidation problems requires novel existence of solutions results for (BS)PDEs with singular terminal condition.

Penalization of open positions at the terminal time is the most common approach in the literature to prove existence of a solution to the HJB equations arising in models of portfolio liquidation. Relaxing the binding liquidation constraint by an quadratic penalization corresponds to a finite terminal condition to the HJB equation. The existence of a minimal solution to the singular equation is then established by monotone approximation as the penalization parameter (the finite terminal value) tends to infinity. Kratz and Schöneborn [KS15] are among the first who applied this *penalization approach* in a framework of optimal liquidation establishing existence to a coupled ODE system with singular terminal condition arising in a multi-asset generalization of the Almgren and Chriss model with active and passive orders without adverse selection. Here, passive orders are orders submitted to a dark pool or crossing network. They do not incur market impact but their execution is uncertain.

Popier [Pop06, Pop07] applied this approximation approach to establish minimal solutions to BSDEs with singular terminal condition. Ankirchner et al. [AJK14] applied, and partially generalized, Popier's work to liquidation problems with non-Markovian parameters. A general class of Markovian liquidation problems has been solved in Schied [Sch13a] by means of Dawson–Watanabe superprocess. This approach avoids the use of HJB equations. Instead, it uses a probabilistic verification argument based on log-Laplace functionals of superprocesses that requires sharp upper and lower bounds for the candidate value function.

In Chapter 2 we apply the penalization approach to a mixed Markov/non-Markov liquidation problem. This chapter is based on Graewe et al. [GHQ15]. This is the first paper to analyze solution concepts for BSPDEs with singular terminal conditions. This paper has recently been generalized by Horst et al. [HQZ16] to the case of degenerate parabolic equations.

In Chapter 3, which is based on [GHS16], we establish a novel *asymptotic approach* to solve HJB equations with singular terminal values. This approach in particular allows us to prove existence of smooth solutions (rather than weak solutions as in Chapter 2). The idea of the asymptotic approach is to determine first the precise asymptotic singular behavior of a potential solution the HJB equation at the terminal time. The asymptotics of the solution educate an asymptotic ansatz that reduces the HJB equation with singular terminal value to a semilinear parabolic PDE with a *finite* terminal condition yet singular nonlinearity.

All the aforementioned liquidation models only allow for purely instantaneous impact ("infinite resilience"). In a second class of models, initiated by Obizhaeva and Wang [OW13], price impact is described within a block shaped limit order book model with *finite* resilience. In such a model the impact is persistent and effects subsequent orders but decays over time. When impact is persistent one often allows for both absolutely continuous and singular trading strategies. In [OW13] the authors assumed constant resilience and market depth. Fruth et al. [FSU14] generalized the model to deterministic time-varying market depths and resiliences and obtained a closed form solution by calculus of variation techniques. In the follow up work [FSU15] the authors allowed for stochastic liquidity parameters. They showed the state space divides into a trade and a no-trade region but did not obtain an explicit description of the boundary. Characterization of optimal strategies results in terms of coupled BSDE systems were obtained by Horst and Naujokat [HN14] for a model of optimal curve following in a two-sided limit order book. An explicit solution of the related free-boundary problem in a model with infinite time horizon and multiplicative price impact has recently been given by Becherer et al. [BBF16].

In Chapter 4, which is based on [GH16], we consider a model with both instantaneous and persistent price impact and stochastic resilience when only absolutely continuous trading strategies are admissible. To the best of our knowledge this is the first time where both types of price impact are considered simultaneously. In this model the value function can be described by a novel three-dimensional system of BSDEs with singular terminal condition in one component, for which we extend the asymptotic approach of Chapter 3.

1.1. Summary of Chapter 2

In Chapter 2 we consider the following non-Markovian stochastic optimal control problem with a terminal state constraint:

$$\begin{aligned}
&V_t(x,y) \\
&= \operatorname*{ess\,inf}_{(\xi,\rho)\in\mathcal{A}(t,x)} E\left[\int_t^T \left\{\eta_s(y_s)|\xi_s|^2 + \lambda_s(y_s)|x_s|^2 + \int_{\mathcal{Z}} \gamma_s(y_s,z)|\rho_s(z)|^2\,\mu(dz)\right\} ds \,\middle|\, \bar{\mathcal{F}}_t\right]
\end{aligned}$$

subject to

$$\begin{cases}
x_s = x - \int_t^s \xi_r\,dr - \int_t^s \int_{\mathcal{Z}} \rho_t(z)\,\pi(dz,dr), \\
x_T = 0, \\
y_s = y + \int_t^s b_r(y_r)\,dr + \int_0^t \bar{\sigma}_r(y_r)\,dB_r + \int_t^s \sigma_r(y_r)\,dW_r.
\end{cases}$$

The real-valued process $(x_s)_{s\in[t,T]}$ is the *state process*; in the portfolio liquidation framework x_t describes the number of shares held at time $s \in [t,T]$. The state process is governed by a pair of *controls* (ξ,ρ) describing the rates at which the portfolio is liquidated actively in the primary market and the passive block trades placed in a dark pool, respectively. Dark pool executions are modeled by a point process $\tilde{J}$ on a nonempty Borel set $\mathcal{Z} \subset \mathbb{R}^l$ with finite characteristic measure $\mu(dz)$ and associated Poisson random measure $\pi(dz,dt)$.

The d-dimensional process $(y_t)_{t\in[0,T]}$ is an uncontrolled *factor process*. The factor process is driven by the independent Wiener processes W and B; the coefficients $b_t(y), \bar{\sigma}_t(y)$ and $\sigma_t(y)$ are $\mathcal{F}$-adapted, where $\{\mathcal{F}_t\}_{t\geq 0}$ denotes the complete filtration generated by W. The basis's filtration carrying B, W, and $\tilde{J}$ satisfying the usual conditions of completeness and right-continuity is denoted by $\{\bar{\mathcal{F}}_t\}_{t\geq 0}$. The set $\mathcal{A}(t,x)$ of *admissible controls* consists of all pairs $(\xi,\rho) \in \mathcal{L}^2_{\bar{\mathcal{F}}}(t,T;\mathbb{R}) \times \mathcal{L}^2_{\bar{\mathcal{F}}}(t,T;L^2(\mathcal{Z}))$ that satisfy almost surely the *terminal state constraint*

$$x_T = 0.$$

In a portfolio liquidation framework the coefficients $\eta_t(y)$ and $\lambda_t(y)$ measure the market impact costs and the investor's desire for early liquidation ("risk aversion"), respectively. The term $\gamma_t(y)$ measures the so-called *slippage* or *adverse selection costs* associated with the execution of dark pool orders.

We solve the control problem by solving the corresponding stochastic HJB equation first introduced by Peng [Pen92] for non-Markovian control problems. In view of the linear-quadratic structure of the control problem standard arguments suggest a multiplicative decomposition of the value function of the form

$$V_t(x,y) = u_t(y)x^2 \quad \text{and} \quad \Psi_t(x,y) = \psi_t(y)x^2$$

for a pair of adapted processes (u,ψ) that satisfies the BSPDE (in a suitable class of stochastic processes)

$$\begin{cases} -du_t(y) = \{\mathcal{L}u_t(y) + \mathcal{M}\psi_t(y) + F(s,y,u_t(y))\}\,dt - \psi_t(y)\,dW_t, \ (t,y) \in [0,T]\times\mathbb{R}^d; \\ \quad u_T(y) = +\infty, \quad y \in \mathbb{R}^d, \end{cases}$$

where, for $a = \frac{1}{2}(\sigma\sigma^{\mathcal{T}} + \bar{\sigma}\bar{\sigma}^{\mathcal{T}})$, the operators $\mathcal{L}$ and $\mathcal{M}$ act on twice, respectively once continuously differentiable functions according to

$$\mathcal{L}u_t(y) = \operatorname{tr}\left((a_t(y)D^2u_t(y)\right) + b_t^{\mathcal{T}}(y)Du_t(y) \quad \text{and} \quad \mathcal{M}\psi_t(y) = \operatorname{tr}\left(D\psi_t(y)\sigma_t^{\mathcal{T}}(y)\right)$$

and the non-linearity $F : [0,T] \times \mathbb{R}^d \times L^0(\mathbb{R}^d) \to \mathbb{R}$ is given by

$$F(t,y,\phi(y)) = \lambda_t(y) - \int_{\mathcal{Z}} \frac{|\phi(y)|^2}{\gamma_t(y,z) + \phi(y)}\,\mu(dz) - \frac{|\phi(y)|^2}{\eta_t(y)}.$$

The preceding BSPDE depends quadratically on $u_t(y)$. Although BSPDEs have been extensively studied in the applied probability and financial mathematics literature, see, e.g., [Ben83, CPY09, DZ13, EK09, MT03, TZ09], no general theory for BSPDEs which are of quadratic growth in u is yet available, not even for finite terminal values. Using recent existence of solutions results for nonlinear BSPDEs [Qiu12, QT12, QW14, YT13] and the Itô-Wentzell formula for distribution-valued processes [Kry12, YT13] we first prove that the BSPDE resulting from a corresponding control problem with finite terminal condition has a sufficiently regular weak solution. Subsequently, we establish a comparison principle from which we deduce that the solution to the BSPDE with infinite terminal value can be obtained by the penalization approach as the limit of an increasing sequence of solutions to BSPDEs with finite terminal conditions.

1.2. Summary of Chapter 3

In Chapter 3 we analyze a Markovian version of the optimal liquidation problem considered in Chapter 2 with general linear-power-law structure. In this case the HJB equation reduces to the following semilinear parabolic equation with singular terminal value:

$$\begin{cases} -\partial_t v(t,y) - \mathcal{L}v(t,y) - F(y, v(t,y)) = 0, & (t,y) \in [0,T) \times \mathbb{R}^d, \\ \lim\limits_{t\to T} v(t,y) = +\infty & \text{locally uniformly on } \mathbb{R}^d, \end{cases}$$

with nonlinearity

$$F(y,v) = \lambda(y) - \frac{|v|^{\beta+1}}{\beta\eta(y)^\beta} + \frac{\theta\gamma(y)v}{\sqrt[\beta]{\gamma(y)^\beta + |v|^\beta}} - \theta v,$$

where $\beta = 1/(p-1)$ and $p > 1$ is the exponent to the power-law instantaneous price impact. Here, the point-process governing dark pool executions is a Poisson process with constant intensity $\theta \geq 0$.

A special case of this optimal liquidation problem has been considered earlier in Ankirchner and Kruse [AK12b] where the singular value function is characterized as a viscosity solution to the HJB equation. Existence of weak solutions and optimal strategies can be inferred from Chapter 2. In Chapter 3 we prove existence of smooth solutions under additional smoothness and boundedness conditions on the cost coefficients which have not yet been established in the literature before.

Using a shifting argument we first establish a novel comparison principle for viscosity sub- and supersolutions to time-homogeneous parabolic PDEs with singular terminal condition. The comparison principle allows us to provide sharp a priori estimates by identifying suitable sub- and supersolutions and to infer the precise singular asymptotic behavior

$$(T-t)^{p-1}v(t,y) = \eta(y) + O(T-t) \quad \text{as } t \to T,$$

of the solution HJB equation. This educates the asymptotic ansatz

$$v(t,y) = \frac{\eta(y)}{(T-t)^{p-1}} + \frac{u(t,y)}{(T-t)^p}, \quad u(t,y) = O((T-t)^2),$$

which reduces the original problem with singular terminal condition to the problem[1]

$$\begin{cases} -\partial_t u = \mathcal{L}u + (T-t)\mathcal{L}\eta + (T-t)^2\lambda - \dfrac{u^2}{\eta(T-t)^2}, & (t,y) \in [0,T) \times \mathbb{R}^d, \\ u(T,y) = 0, & y \in \mathbb{R}^d, \end{cases}$$

[1] For simplicity we are here giving the formula for the case $p = 2$ and $\theta = 0$.

with *finite* terminal condition yet singular nonlinearity, for which existence of a unique classical solution can be proved using standard fixed point arguments in a suitable weighted function space.

Using Krylov's generalized Itô formula we prove that the classical solution is indeed the value function and give optimal trading strategies in closed form. Our comparison principle implies uniqueness of a continuous viscosity solution of polynomial growth to the HJB equation under continuity and polynomial growth conditions on the cost coefficients. The shifting argument used in the proof of the comparison principle also allows us to prove that the minimal nonnegative solution to the stochastic HJB equation in [AJK14] is indeed the unique nonnegative solution to their singular BSDE if the coefficients are essentially bounded.

1.3. Summary of Chapter 4

In Chapter 4 we address optimal liquidation problems with both instantaneous and persistent price impact and stochastic non-Markovian resilience when only absolutely continuous trading strategies are admissible. Specifically, we consider the following linear-quadratic non-Markovian stochastic control problem

$$\operatorname*{ess\,inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E}\left[\int_0^T \tfrac{1}{2}\eta\xi_s^2 + \xi_s Y_s + \tfrac{1}{2}\lambda_s X_s^2 \, ds\right]$$

subject to

$$\begin{cases} X_t = x - \displaystyle\int_0^t \xi_s \, ds, \quad t \in [0,T], \\ X_T = 0, \\ Y_t = y + \displaystyle\int_0^t -\rho_s Y_s + \gamma\xi_s \, ds, \quad t \in [0,T]. \end{cases}$$

Here, η and γ are positive constants and ρ and λ are progressively measurable, non-negative and essentially bounded stochastic processes:

$$\eta > 0, \gamma \in \mathbb{R}_+; \quad \rho, \lambda \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}_+).$$

In the portfolio liquidation framework the process Y describes the persistent price impact caused by past trades in a block-shaped limit order book market with constant order book depth $1/\gamma > 0$ as in Obizhaeva and Wang [OW13]. One interpretation is that the trading rate ξ adds a drift to an underlying fundamental martingale price process. The stochastic process $\rho \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}_+)$ describes the rates at which the order book recovers from past trades. The constant $\eta > 0$ describes an additional instantaneous impact factor and λ the risk aversion as in Chapters 2 and 3. The spacial case $\rho \equiv 0$, $y = 0$, and $\lambda \equiv$ *const.* corresponds to model of Almgren and Chriss [AC01].

The restriction to absolutely continuous strategies allows us to formulate the resulting control problem within in a classical, rather than singular stochastic control framework, and to obtain a closed form solution for both, the value function and the optimal trading strategy. Characterizing the value function is typically hard if singular controls are allowed. In fact, when both absolutely continuous and singular controls are admissible as in e.g. [HN14], one typically only obtains characterization results for optimal controls using maximum principles.

Within our modeling framework, the value function can be represented in terms of the solution to a fully coupled *three-dimensional* stochastic Riccati equation (BSDE system). For the benchmark case of constant model parameters the stochastic system reduces to a deterministic ODE system. For this case we illustrate how our model can be used to approximate liquidation models with block trades and can, hence, be viewed as a first step towards a unified approach to singular and regular stochastic control problems with singular terminal values.

In proving the existence of a unique solution to the BSDE system two main challenges need to be overcome. First, the liquidation constraint imposes a singular terminal condition on the first component of the BSDE system. Second, our BSDE system does not satisfy the quasi-monotonicity condition that is necessary for the multi-dimensional comparison principle in [HP06] to hold.

The idea is to extend the asymptotic approach introduced in Chapter 3 to the BSDE system and to characterize the solution to the BSDE system with singular terminal value in terms of a BSDE with finite terminal value yet singular driver. Finally, we establish the verification result from which we deduce uniqueness of solutions to the BSDE system as well as a closed-form representation of the optimal trading strategy.

Establishing the a priori estimates for our BSDE system is key for both the proof of existence of a solution and the verification theorem. As pointed out above the BSDE system that characterizes the value does not satisfy the quasi-monotonicity condition of Hu and Peng [HP06]. In order to overcome this problem we consider the joint dynamics of the BSDE that describes the value function and two additional BSDEs that describe the candidate optimal trading strategy. Using the comparison principle for BSDE systems in [HP06] we first determine the range of all these processes from which we then deduce the desired deterministic upper bounds for the coefficients of the value function.

2. A Non-Markovian Liquidation Problem and Backward SPDEs with Singular Terminal Conditions

2.1. Model and problem formulation

Throughout this chapter, we work on a probability space $(\Omega, \bar{\mathcal{F}}, \mathbb{P})$ equipped with a filtration $\{\bar{\mathcal{F}}_t\}_{0\leq t\leq T}$ that satisfies the usual conditions of completeness and right-continuity. The probability space carries two independent m-dimensional[1] Brownian motions W and B as well as an independent point process $\tilde{J}$ on on a non-empty Borel set $\mathcal{Z} \subset \mathbb{R}^l$ with finite characteristic measure $\mu(dz)$. We endow the set $\mathcal{Z}$ with its Borel σ-algebra $\mathscr{Z}$ and denote by $\pi(dz, dt)$ the associated Poisson random measure. The filtration generated by W, together with all $\mathbb{P}$ null sets, is denoted by $\{\mathcal{F}_t\}_{t\geq 0}$. The σ-algebra of the predictable sets on $\Omega \times [0, +\infty)$ associated with $\{\mathcal{F}_t\}_{t\geq 0}$ is denoted by $\mathscr{P}$.

In this chapter, we address the following stochastic optimal control problem with a terminal state constraint:

$$\min_{\xi,\rho} E\left[\int_0^T \left\{\eta_s(y_s)|\xi_s|^2 + \lambda_s(y_s)|x_s|^2 + \int_{\mathcal{Z}} \gamma_s(y_s, z)|\rho_s(z)|^2\, \mu(dz)\right\} ds\right]$$

subject to

$$\begin{cases} x_t = x - \int_0^t \xi_s\, ds - \int_0^t \int_{\mathcal{Z}} \rho_s(z)\, \pi(dz, ds), \quad t \in [0, T]; \\ x_T = 0; \\ y_t = y + \int_0^t b_s(y_s)\, ds + \int_0^t \bar{\sigma}_s(y_s)\, dB_s + \int_0^t \sigma_s(y_s)\, dW_s. \end{cases}$$

The real-valued process $(x_t)_{t\in[0,T]}$ is the *state process*; in a portfolio liquidation framework x_t describes the number of shares held at time $t \in [0, T]$. The state process is governed by a pair of *controls* (ξ, ρ) describing, for instance, the rates at which the portfolio is liquidated in the primary market and the block trades placed in the dark pool, respectively, with the Poisson random measure π governing dark pool executions.

The d-dimensional process $(y_t)_{t\in[0,T]}$ is an uncontrolled *factor process*. The factor process is driven by the Wiener processes W and B; the coefficients $b_t(y), \bar{\sigma}_t(y)$

[1] The Brownian motions may well have different dimensions; this assumption is made for convenience only.

and $\sigma_t(y)$ are $\mathcal{F}$-adapted. We sometimes write $x_s^{t,x,\xi,\rho}$ for $0 \le t \le s \le T$ to indicate the dependence of the state process on the control (ξ, ρ), the initial time $t \in [0,T]$ and initial state $x \in \mathbb{R}$. Likewise, we sometimes write $y_s^{t,y}$. The set $\mathcal{A}(t,x)$ of *admissible controls* at state $(t,x) \in [0,T) \times \mathbb{R}$ consists of all pairs $(\xi,\rho) \in \mathcal{L}^2_{\bar{\mathcal{F}}}(t,T;\mathbb{R}) \times \mathcal{L}^2_{\bar{\mathcal{F}}}(t,T;L^2(\mathcal{Z}))$ that satisfy almost surely the *terminal state constraint*

$$x_T = 0. \tag{2.1}$$

We assume that the cost associated with an admissible control (ξ,ρ) at time $t \in [0,T)$ and state $(x,y) \in \mathbb{R} \times \mathbb{R}^d$ is given by

$$J_t(x,y;\xi,\rho) :=$$
$$E\left[\int_t^T \left\{ \eta_s(y_s^{t,y})|\xi_s|^2 + \lambda_s(y_s^{t,y})|x_s^{t,x;\xi,\rho}|^2 + \int_{\mathcal{Z}} \gamma_s(y_s^{t,y},z)|\rho_s(z)|^2\,\mu(dz) \right\} ds \,\middle|\, \bar{\mathcal{F}}_t \right]$$

for $\mathcal{F}$-adapted coefficients $\eta_t(y)$, $\lambda_t(y)$ and $\gamma_t(y)$. The *value function* is denoted by

$$V_t(x,y) := \operatorname*{ess\,inf}_{(\xi,\rho)\in\mathcal{A}(t,x)} J_t(x,y;\xi,\rho) \tag{2.2}$$

In a portfolio liquidation framework the coefficients $\eta_t(y)$ and $\lambda_t(y)$ measure the market impact costs and the investor's desire for early liquidation ("risk aversion"), respectively. The term $\gamma_t(y)$ measures the so-called *slippage* or *adverse selection costs* associated with the execution of dark pool orders.[2] $V_t(x,y)$ is the cost of liquidating the portfolio comprising x shares during the time interval $[t,T]$, given the current value y of the factor process and (2.1) reflects the fact that full liquidation is required by the terminal time.

2.1.1. The BSPDE for the value function

We solve the control problem by solving the corresponding stochastic HJB equation for non-Markovian control problems. In view of the linear-quadratic structure of the cost functional a standard arguments suggest a multiplicative decomposition of the value function of the form

$$V_t(x,y) = u_t(y)x^2 \quad \text{and} \quad \Psi_t(x,y) = \psi_t(y)x^2 \tag{2.3}$$

for a pair of adapted processes (u,ψ) that satisfies the BSPDE (in a suitable class of stochastic processes)

$$\begin{cases} -du_t(y) = \{\mathcal{L}u_t(y) + \mathcal{M}\psi_t(y) + F(s,y,u_t(y))\}\,dt \\ \qquad\qquad - \psi_t(y)\,dW_t, \quad (t,y) \in [0,T] \times \mathbb{R}^d; \\ u_T(y) = +\infty, \quad y \in \mathbb{R}^d, \end{cases} \tag{2.4}$$

[2]The notion of "slippage costs" refers to the costs associated with an adversely executed order, e.g., a buy order execution in a dark pool immediately before a price decrease in the primary market.

where, for $a := \frac{1}{2}(\sigma\sigma^{\mathcal{T}} + \bar{\sigma}\bar{\sigma}^{\mathcal{T}})$, the operators $\mathcal{L}$ and $\mathcal{M}$ act on twice, respectively once continuously differentiable functions according to

$$\mathcal{L}u_t(y) = \mathrm{tr}\left((a_t(y)D^2u_t(y)\right) + b_t^{\mathcal{T}}(y)Du_t(y) \quad \text{and} \quad \mathcal{M}\psi_t(y) = \mathrm{tr}\left(D\psi_t(y)\sigma_t^{\mathcal{T}}(y)\right)$$

with D and D^2 being the gradient and Hessian operator, respectively, throughout this chapter, and the non-linearity $F : [0,T] \times \mathbb{R}^d \times L^0(\mathbb{R}^d) \to \mathbb{R}$ is given by

$$F(t,y,\phi(y)) := \lambda_t(y) - \int_{\mathcal{Z}} \frac{|\phi(y)|^2}{\gamma_t(y,z) + \phi(y)}\,\mu(dz) - \frac{|\phi(y)|^2}{\eta_t(y)}.$$

The remainder of this chapter is organized as follows. Our main assumptions and results are summarized in Section 2. Section 3 is devoted to the proof of the verification theorem while Section 4 establishes the existence of the solution for our singular BSPDE that satisfies the assumptions of the verification theorem. In Section 5 we prove that the BSPDE (2.4) actually has a unique non-negative solution in a larger class of stochastic processes that automatically satisfies the asymptotic behavior around the terminal time that is needed for the proof of the verification theorem. The appendix recalls three results on BSPDEs which are used throughout this chapter.

2.2. The main results

In order to state our main result we need to introduce some function spaces. For a Banach space V we denote by $\mathcal{S}^p_{\mathcal{F}}([0,T];V)$, $p \in [1,\infty)$, the set of all the V-valued and $\mathscr{P}$-measurable càdlàg processes $(X_t)_{t\in[0,T]}$ such that

$$\|X\|^p_{\mathcal{S}^p_{\mathcal{F}}([0,T];V)} = E\left[\sup_{t\in[0,T]} \|X_t\|^p_V\right] < \infty.$$

By $\mathcal{L}^p_{\mathcal{F}}(0,T;V)$ we denote the class of V-valued $\mathscr{P}$-measurable processes $(u_t)_{t\in[0,T]}$ such that

$$\|u\|^p_{\mathcal{L}^p_{\mathcal{F}}(0,T;V)} = E\left[\int_0^T \|u_t\|^p_V\,dt\right] < \infty, \quad p \in [1,\infty);$$
$$\|u\|_{\mathcal{L}^\infty_{\mathcal{F}}(0,T;V)} = \operatorname*{ess\,sup}_{(\omega,t)\in\Omega\times[0,T]} \|u_t\|_V < \infty, \quad p = \infty.$$

Similarly we define $\mathcal{S}^p_{\bar{\mathcal{F}}}([0,T];V)$ and $\mathcal{L}^p_{\bar{\mathcal{F}}}(0,T;V)$. For $u \in \mathcal{L}^p_{\mathcal{F}}(0,T;L^p(\mathbb{R}^d))$, $p \in [1,\infty)$, we write $u \in \mathcal{L}^{p,\infty}_{\mathcal{F}}(0,T)$ if

(i) u is continuous on $[0,T]$, $\mathbb{P}\otimes dx$-a.e.;

(ii) $\|u\|^p_{\mathcal{L}^{p,\infty}_{\mathcal{F}}(0,T)} = E\int_{\mathbb{R}^d} \sup_{t\in[0,T]} |u(t,x)|^p\,dx < \infty.$

As usual, the Sobolev space of all functions whose first k derivatives belong to $L^p(\Pi)$ for some domain $\Pi \subset \mathbb{R}^d$ is denoted by $H^{k,p}(\Pi)$. For simplicity, by saying a finite dimensional space-valued function $u = (u_1, \dots, u_l) \in H^{k,p}(\Pi)$, $l \in \mathbb{N}$, we mean $u_1, \dots, u_l \in H^{k,p}(\Pi)$ and $\|u\|^p_{H^{k,p}(\Pi)} := \sum_{j=1}^{l} \|u_j\|^p_{H^{k,p}(\Pi)}$.

Throughout this chapter, we use $\langle \cdot, \cdot \rangle$ to denote the inner product in the usual Hilbert space $L^2(\mathbb{R}^d) = H^{0,2}(\mathbb{R}^d)$. For $k \in \mathbb{N}_0$, we set

$$\mathcal{H}^k = \mathcal{S}^2_{\mathcal{F}}([0,T]; H^{k,2}(\mathbb{R}^d)) \cap \mathcal{L}^2_{\mathcal{F}}(0,T; H^{k+1,2}(\mathbb{R}^d))$$

equipped with the norm

$$\|u\|^2_{\mathcal{H}^k} = \|u\|^2_{\mathcal{S}^2_{\mathcal{F}}([0,T];H^{k,2}(\mathbb{R}^d))} + \|u\|^2_{\mathcal{L}^2_{\mathcal{F}}(0,T;H^{k+1,2}(\mathbb{R}^d))}.$$

Our goal is to prove existence of a sufficiently regular solution to the BSPDE (2.4) and to characterize the value function of our control problem in terms of that solution. To this end, we first define what we mean by a solution to (2.4).

Definition 2.2.1. A pair of processes (u, ψ) is a solution to the BSPDE (2.4) if for all $0 \le t < \tau < T$ it holds $(u, \psi)1_{[0,\tau]\times\mathcal{O}} \in \mathcal{L}^2_{\mathcal{F}}(0,\tau; H^{2,2}(\mathcal{O})) \times \mathcal{L}^2_{\mathcal{F}}(0,\tau; H^{1,2}(\mathcal{O}))$ for all bounded balls $\mathcal{O} \subset \mathbb{R}^d$,

$$u_t(y) = u_\tau(y) + \int_t^\tau \{\mathcal{L}u_s(y) + \mathcal{M}\psi_s(y) + F(s, y, u_s(y))\}\, ds - \int_t^\tau \psi_s(y)\, dW_t, \quad dy\text{-a.e.},$$

and

$$\lim_{\tau \uparrow T} u_\tau(y) = +\infty, \quad \mathbb{P} \otimes dy\text{-a.e.}$$

Our results are established under the following standard measurability and regularity conditions on the model parameters:

($\mathcal{A}1$) The function

$$(b, \sigma, \bar{\sigma}, \eta, \lambda) : \Omega \times [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d\times m} \times \mathbb{R}^{d\times m} \times \mathbb{R}_+ \times \mathbb{R}_+$$

is $\mathscr{P} \times \mathscr{B}(\mathbb{R}^d)$-measurable and essentially bounded by $\Lambda > 0$. Moreover,

$$\gamma : \Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{Z} \longrightarrow [0, +\infty],$$

is $\mathscr{P} \times \mathscr{B}(\mathbb{R}^d) \times \mathscr{Z}$-measurable.

($\mathcal{A}2$) There exists a constant L such that for all $y_1, y_2 \in \mathbb{R}^d$ and $(\omega, t) \in \Omega \times [0,T]$,

$$|b_t(y_1) - b_t(y_2)| + |\sigma_t(y_1) - \sigma_t(y_2)| + |\bar{\sigma}_t(y_1) - \bar{\sigma}_t(y_2)| \le L|y_1 - y_2|.$$

($\mathcal{A}3$) There exist positive constants κ and κ_0 such that for all $(y, \xi, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0,T]$,

$$\sum_{i,j=1}^{d} \sum_{r=1}^{m} \bar{\sigma}^{ir}_t(y)\bar{\sigma}^{jr}_t(y)\xi^i\xi^j \ge \kappa|\xi|^2 \quad \text{and} \quad \eta_t(y) \ge \kappa_0, \quad \mathbb{P}\text{-a.e.}$$

The verification theorem requires an integral representation of the process

$$\left\{u_t(y_t^{0,y})|x_t^{0,x,\xi,\rho}|^2\right\}_{0\le t\le T}. \tag{2.5}$$

We are unaware of a general L^∞-theory for BSPDEs; at the same time, under assumptions $(\mathcal{A}1)-(\mathcal{A}3)$, we can not apply the existing L^p-theory ($p\in(1,\infty)$) in our framework directly; see [DQT12] and references therein. Moreover, as it will turn out, the solution u to (2.4) has to be regular enough to allow for an application of the generalized Itô-Kunita-Wentzell formula of Yang and Tang [YT13] to the composition $u_t(y_t)$. To guarantee regularity and apply the existing L^p-theory on BSPDEs, we work with a weighted solution. More precisely, we define, for any integer $q>d$, the function

$$\theta:\mathbb{R}^d\to\mathbb{R},\quad y\mapsto(1+|y|^2)^{-q},$$

and analyze θu instead of u. A direct computation verifies that (u,ψ) is a solution to (2.4) if and only if $(\theta u,\theta\psi)$ solves

$$\begin{cases} -dv_t(y)=\{\tilde{\mathcal{L}}v_t(y)+\tilde{\mathcal{M}}\zeta_t(y)+\theta F(t,y,(\theta^{-1}v_t)(y))\}\,dt \\ \qquad\qquad -\zeta_t(y)\,dW_t,\quad (t,y)\in[0,T)\times\mathbb{R}^d; \\ v_T(y)=+\infty,\quad y\in\mathbb{R}^d, \end{cases} \tag{2.6}$$

where

$$\tilde{\mathcal{L}}v_t(y):=\operatorname{tr}(a_t(y)D^2v_t(y))+\tilde{b}_t^{\mathcal{T}}(y)Dv_t(y)+c_t(y)v_t(y)$$

and

$$\tilde{\mathcal{M}}\zeta_t(y):=\operatorname{tr}(D\zeta_t(y)\sigma_t^{\mathcal{T}}(y))+\beta_t^{\mathcal{T}}(y)\zeta_t(y)$$

and the functions $\tilde{b}_t=(\tilde{b}_t^i)_{i=1}^d$, $\beta_t=(\beta_t^r)_{r=1}^m$ and c_t are given by

$$\begin{aligned} \tilde{b}_t^i(y):&=b_t^i(y)+\frac{4q}{1+|y|^2}\sum_{j=1}^d a_t^{ij}(y)y^j,\\ \beta_t^r(y):&=\frac{2q}{1+|y|^2}\sum_{j=1}^d\sigma_t^{jr}(y)y^j,\\ c_t(y):&=\frac{2q}{1+|y|^2}\left(\operatorname{tr}(a_t(y))+\sum_{i=1}^d y^ib_t^i(y)+\frac{2(q-1)}{1+|y|^2}\sum_{i,j=1}^d a_t^{ij}(y)y^iy^j\right). \end{aligned}$$

For each $\delta\in(0,1)$, let $C^\delta(\mathbb{R}^d)$ be the usual Hölder space on $\mathbb{R}^d$. We are now ready to summarize the main results of this chapter.

Theorem 2.2.2. *Under assumptions $(\mathcal{A}1)-(\mathcal{A}3)$ the following holds:*

(i) The BSPDE (2.4) admits a solution (u,ψ) which satisfies

$$(\theta u,\theta\psi)1_{[0,\tau]}\in\mathcal{H}^1\times\mathcal{L}^2_{\mathcal{F}}(0,T;H^{1,2}(\mathbb{R}^d)),\quad \tau\in[0,T), \tag{2.7}$$

and

$$\frac{c_0}{T-t} \leq u_t(y) \leq \frac{c_1}{T-t}, \quad \mathbb{P} \otimes dt \otimes dy\text{-a.e.},$$

with c_0 and c_1 being two positive constants. The function

$$V(t, y, x) := u_t(y)x^2, \quad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d, \tag{2.8}$$

coincides with the value function for almost every $y \in \mathbb{R}^d$ and the optimal (feedback) control is given by

$$(\xi_t^*, \ \rho_t^*(z)) = \left(\frac{u_t(y_t)x_t}{\eta_t(y_t)}, \ \frac{u_t(y_t)x_{t-}}{\gamma_t(z, y_t) + u_t(y_t)} \right).$$

(ii) The solution (u, ψ) is the unique non-negative solution to (2.4) in that sense that if $(\bar{u}, \bar{\psi})$ is another solution satisfying (2.7) and $\bar{u} \geq 0$, $\mathbb{P} \otimes dt \otimes dx$-a.e., then

$$\bar{u}_t(y) = u_t(y), \quad \mathbb{P} \otimes dt \otimes dy\text{-a.e.}$$

(iii) Under the additional assumption that σ is spatially invariant, i.e., does not depend on y one has furthermore for any $p \in (2, +\infty)$,

$$\theta(\cdot)u_\cdot\left(\cdot + \int_0^\cdot \sigma_s \, dW_s \right) \in \bigcap_{\tau \in (0,T)} \bigcap_{\delta \in (0,1)} \mathcal{L}_{\mathcal{F}}^{2,\infty}(0, \tau) \cap \mathcal{S}_{\mathcal{F}}^p([0, \tau]; C^\delta(\mathbb{R}^d))$$

and the function $V(t, y, x)$ in (2.8) coincides with the value function for every $y \in \mathbb{R}^d$.

Remark 2.2.3. When all the coefficients $b, \sigma, \bar{\sigma}, \lambda, \eta, \gamma$ are deterministic functions, then the optimal control problem is Markovian and the corresponding BSPDE (2.4) reduces to a deterministic parabolic partial differential equation

$$\begin{cases} -\partial_t u = \mathcal{L}u + F(t, y, u), & (t, y) \in [0, T] \times \mathbb{R}^d; \\ u_T(y) = +\infty, & y \in \mathbb{R}^d. \end{cases} \tag{2.9}$$

In this case, we may with no loss of generality assume that $\sigma \equiv 0$ so Theorem 2.2.2 (iii) indicates that (2.9) admits a unique non-negative solution u in the distributional sense that satisfies

$$\theta u \in \bigcap_{\tau \in (0,T)} \bigcap_{\delta \in (0,1)} C([0, \tau]; C^\delta(\mathbb{R}^d)),$$

and $u_t(y)x^2$ coincides with the continuous value function for every $y \in \mathbb{R}^d$.

2.3. The verification theorem

We are now ready to state the verification theorem. Its proof requires some preparation and is carried out below.

Theorem 2.3.1. *Let assumptions* $(\mathcal{A}1)-(\mathcal{A}3)$ *be satisfied and suppose that* (u,ψ) *is a solution to* (2.4) *that satisfies*

$$(\theta u, \theta\psi)1_{[0,t]} \in \mathcal{H}^1 \times \mathcal{L}^2_{\mathcal{F}}(0,T;H^{1,2}(\mathbb{R}^d)), \quad t\in[0,T), \tag{2.10}$$

and

$$\frac{c_0}{T-t} \le u_t(y) \le \frac{c_1}{T-t}, \quad \mathbb{P}\otimes dt\otimes dy\text{-a.e.}, \tag{2.11}$$

with c_0 *and* c_1 *being two positive constants. Then,* $\theta u \in \cap_{\tau\in(0,T)}\mathcal{L}^{2,\infty}_{\mathcal{F}}(0,\tau)$ *and*

$$V(t,y,x) := u_t(y)x^2, \quad (t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d,$$

coincides with the value function of (2.2) *for almost every* $y\in\mathbb{R}^d$*. Moreover, the optimal (feedback) control is given by*

$$(\xi^*_t,\, \rho^*_t(z)) = \left(\frac{u_t(y_t)x_t}{\eta_t(y_t)},\, \frac{u_t(y_t)x_{t-}}{\gamma_t(z,y_t)+u_t(y_t)}\right).$$

We first recall the following generalized Itô-Kunita-Wentzell formula from which we later derive an integral representation for (2.5).

Lemma 2.3.2 ([YT13, Theorem 3.1])**.** *Let the coefficients* b*,* σ*, and* $\bar\sigma$ *satisfy* $(\mathcal{A}1)-(\mathcal{A}3)$ *and let* $G\in L^2(\Omega,\mathcal{F}_T;H^{1,2}(\mathbb{R}^d))$*,* $\Phi\in\mathcal{L}^2_{\mathcal{F}}(0,T;H^{2,2}(\mathbb{R}^d))$*,* $\Upsilon\in\mathcal{L}^2_{\mathcal{F}}(0,T;H^{1,2}(\mathbb{R}^d))$ *and* $F\in\mathcal{L}^2_{\mathcal{F}}(0,T;L^2(\mathbb{R}^d))$ *such that*

$$\Phi_t(y) = G(y) + \int_t^T F_s(y)\,ds - \int_t^T \Upsilon_s(y)\,dW_s, \quad dy\text{-a.e.}, \quad \text{for all } t\in[0,T].$$

Then, the compositions $\Phi_\cdot(y^{s,\cdot}_\cdot)$*,* $G(y^{s,\cdot}_T)$*,* $F_\cdot(y^{s,\cdot}_\cdot)$ *and* $\Upsilon_\cdot(y^{s,\cdot}_\cdot)$ *are well-defined under the measure* $\mathbb{P}\otimes dt\otimes dy$*, and for almost every* $y\in\mathbb{R}^d$ *it holds almost surely in* $[0,T]$*,*

$$\begin{aligned}\Phi_t(y^{s,y}_t) = G(y^{s,y}_T) - \int_t^T &\Big\{ \operatorname{tr}\left(a_r(y^{s,y}_r)D^2\Phi_r(y^{s,y}_r) + D\Upsilon_r(y^{s,y}_r)\sigma^{\mathcal{T}}_r(y^{s,y}_r)\right) \\ &\quad + b^{\mathcal{T}}_r(y^{s,y}_r)D\Phi_r(y^{s,y}_r) - F_r(y^{s,y}_r)\Big\}\,dr \\ - \int_t^T &\left\{\sigma^{\mathcal{T}}_r(y^{s,y}_r)D\Phi_r(y^{s,y}_r) + \Upsilon_r(y^{s,y}_r)\right\}dW_r - \int_t^T \bar\sigma^{\mathcal{T}}_r(y^{s,y}_r)D\Phi_r(y^{s,y}_r)\,dB_r.\end{aligned}$$

Using local estimates for the weak solutions to BSPDEs from [QT12], Yang and Tang [YT13] proved that the above compositions are well defined. But they did not establish the integrability properties needed for our proof of the verification theorem. The following corollary establishes such properties.

Corollary 2.3.3. *Under the hypothesis of Lemma 2.3.2, $\Phi_\cdot(y_\cdot^{s,y})$ is a continuous and uniformly integrable semi-martingale for almost every $y \in \mathbb{R}^d$ and $\Phi \in \mathcal{L}^{2,\infty}_{\mathcal{F}}(0,T)$. Furthermore, there exists a constant C that depends only on κ, L, Λ and T such that*

(i) $\displaystyle\int_{\mathbb{R}^d} E[|G(y_T^{s,y})|]^2 dy \leq C\|G\|^2_{L^2(\Omega,\mathcal{F}_T;H^{1,2}(\mathbb{R}^d))}$;

(ii) $\displaystyle\int_{\mathbb{R}^d}\left(\int_s^T E[|F_r(y_r^{s,y})|]\,dr\right)^2 dy \leq C\|F\|^2_{\mathcal{L}^2_{\mathcal{F}}(s,T;L^2(\mathbb{R}^d))}$;

(iii) $\displaystyle\int_{\mathbb{R}^d}\left(\sup_{r\in[s,T]} E[|\Phi_r(y_r^{s,y})|]\right)^2 dy \leq C\big(\|G\|^2_{L^2(\Omega,\mathcal{F}_T;H^{1,2}(\mathbb{R}^d))}+\|F\|^2_{\mathcal{L}^2_{\mathcal{F}}(s,T;L^2(\mathbb{R}^d))}\big)$.

Proof. The process Φ can be seen as an $L^2(\mathbb{R}^d)$-valued continuous semi-martingale. Thus, $\Phi \in \mathcal{H}^0$ and we can further verify that

$$(\Phi,\Upsilon) \in \big(\mathcal{H}^0 \cap \mathcal{L}^2_{\mathcal{F}}(0,T;H^{2,2}(\mathbb{R}^d))\big) \times \mathcal{L}^2_{\mathcal{F}}(0,T;H^{1,2}(\mathbb{R}^d))$$

satisfies (A.1) with

$$f(t,y) := F_t(y) - \left(\operatorname{tr}\left(a_t(y)D^2\Phi_t(y) + D\Upsilon_t(y)\sigma_t^{\mathcal{T}}(y)\right) + b_t^{\mathcal{T}}(y)D\Phi_t(y)\right).$$

Thus, $\Phi \in \mathcal{L}^{2,\infty}_{\mathcal{F}}(0,T)\cap\mathcal{H}^1$ by Proposition A.1.1. For each $N \in \mathbb{N}$, let $(u^N,\psi^N) \in \mathcal{H}^1 \times \mathcal{L}^2_{\mathcal{F}}(0,T;H^{1,2}(\mathbb{R}^d))$ be the unique solution to

$$\begin{cases} -du_t^N(y) = \left\{\operatorname{tr}\big(a_t(y)D^2u_t^N(y) + D\psi_t^N(y)\sigma_t^{\mathcal{T}}(y)\big) + b_t^{\mathcal{T}}(y)Du_t^N(y) + N\wedge|F_t(y)|\right\}dt \\ \qquad\qquad\quad - \psi_t^N(y)\,dW_t, \quad (t,y)\in[0,T]\times\mathbb{R}^d; \\ u_T^N(y) = N\wedge|G(y)|, \quad y\in\mathbb{R}^d. \end{cases}$$

By Lemma 2.3.2, we have for almost every $y \in \mathbb{R}^d$,

$$u_t^N(y_t^{s,y}) = N\wedge|G(y_T^{s,y})| + \int_t^T N\wedge|F_r(y_r^{s,y})|\,dr$$
$$- \int_t^T \left\{\sigma_r^{\mathcal{T}}(y_r^{s,y})Du_r^N(y_r^{s,y}) + \psi_r^N(y_r^{s,y})\right\}dW_r - \int_t^T \bar{\sigma}_r^{\mathcal{T}}(y_r^{s,y})Du_r^N(y_r^{s,y})\,dB_r,$$

where all the compositions are well defined under the measure $\mathbb{P}\otimes dt\otimes dy$. In particular,

$$u_s^N(y) = E\left[N\wedge|G(y_T^{s,y})| + \int_s^T N\wedge|F_r(y_r^{s,y})|\,dr\middle|\mathcal{F}_s\right],$$

while Proposition A.1.1 yields a constant C depending only on κ, L, Λ and T such that

$$\begin{aligned}&\|u^N\|_{\mathcal{L}^{2,\infty}_{\mathcal{F}}(s,T)} + \|\psi^N\|_{\mathcal{L}^2_{\mathcal{F}}(s,T;H^{1,2}(\mathbb{R}^d))} \\ &\leq C\left(\|N\wedge|F|\|_{\mathcal{L}^2_{\mathcal{F}}(0,T;L^2(\mathbb{R}^d))} + \|N\wedge|G|\|_{L^2(\Omega,\mathcal{F}_T;H^{1,2}(\mathbb{R}^d))}\right) \\ &\leq C\left(\|F\|_{\mathcal{L}^2_{\mathcal{F}}(0,T;L^2(\mathbb{R}^d))} + \|G\|_{L^2(\Omega,\mathcal{F}_T;H^{1,2}(\mathbb{R}^d))}\right).\end{aligned}$$

Letting $N \to \infty$, by Fatou's lemma and Jensen's inequality, we obtain

$$\int_{\mathbb{R}^d} \left(E[|G(y_T^{s,y})|] + \int_s^T E[|F_r(y_r^{s,y})|]\, dr \right)^2 dy \\ \leq C \left(\|G\|^2_{L^2(\Omega,\mathcal{F}_T;H^{1,2}(\mathbb{R}^d))} + \|F\|^2_{\mathcal{L}^2_{\mathcal{F}}(s,T;L^2(\mathbb{R}^d))} \right).$$

This proves the desired estimates as well as the fact that $\Phi_\cdot(y_\cdot^{s,y})$ is a continuous and uniformly integrable semi-martingale for almost every $y \in \mathbb{R}^d$. $\square$

Our second auxiliary result is the following lemma on the set of admissible controls. It states that we may with no loss of generality assume that the state process associated with an admissible control is monotone. A similar result has been established in [GHS16] for the Markovian case.

Lemma 2.3.4. *For each admissible control (ξ, ρ) there exists a corresponding admissible control $(\hat{\xi}, \hat{\rho})$ with lesser or equal cost such that the process $x^{0,x;\hat{\xi},\hat{\rho}}$ is almost surely monotone. Furthermore, there exists a constant $C < \infty$ which is independent of $t, x, \hat{\rho}, \hat{\xi}$ such that*

$$|x_t^{0,x;\hat{\xi},\hat{\rho}}|^2 \leq C(T-t)E\left[\int_t^T |\hat{\xi}_s|^2\, ds \middle| \bar{\mathcal{F}}_t \right], \quad t \in [0,T]. \tag{2.12}$$

Proof. Assume that $x \geq 0$ (the case for $x \leq 0$ follows in a similar way). For the admissible control (ξ, ρ), let $(\tilde{x}_t) \in \mathcal{S}^2_{\mathcal{F}}([0,T])$ be the unique solution of the following stochastic differential equation

$$\tilde{x}_t = x - \int_0^t \xi_s^+\, ds - \int_0^t \int_{\mathcal{Z}} \rho_s^+(z) \wedge \tilde{x}_s^+\, \pi(dz, ds),$$

where $f^+ := \max\{f, 0\}$ for $f = \tilde{x}_s, \xi_s$ or ρ_s. Set

$$\hat{\xi}_t := \xi_t^+ 1_{\tilde{x}_t > 0} \quad \text{and} \quad \hat{\rho}_t(z) := \rho_t^+(z) \wedge \tilde{x}_s^+.$$

It is easy to check that $(\hat{\xi}, \hat{\rho}) \in \mathcal{L}^2_{\bar{\mathcal{F}}}(0,T) \times \mathcal{L}^2_{\bar{\mathcal{F}}}(0,T;L^2(\mathcal{Z}))$ is an admissible control pair with lesser or equal cost and that $x^{0,x;\hat{\xi},\hat{\rho}}$ is decreasing almost surely. Since $x^{0,x;\hat{\xi},\hat{\rho}}$ is non-negative and decreasing,

$$0 \leq \hat{\rho}_t \leq x_t^{0,x;\hat{\xi},\hat{\rho}}, \quad \mathbb{P} \otimes dt \otimes \mu(dz)\text{-a.e.}$$

Thus,

$$|x_t^{0,x;\hat{\xi},\hat{\rho}}|^2 \leq 2E\left[\left| \int_t^T \hat{\xi}_s\, ds \right|^2 + \left| \int_{[t,T]\times\mathcal{Z}} \hat{\rho}_{s-}(z)\, \pi(dz, ds) \right|^2 \middle| \bar{\mathcal{F}}_t \right] \\ \leq C(T-t)E\left[\int_t^T |\hat{\xi}_s|^2\, ds + \int_t^T |x_s^{0,x;\hat{\xi},\hat{\rho}}|^2\, ds \middle| \bar{\mathcal{F}}_t \right],$$

which by Gronwall's inequality implies

$$|x_t^{0,x;\hat{\xi},\hat{\rho}}|^2 \le C(T-t)E\left[\int_t^T |\hat{\xi}_s|^2\,ds \middle| \bar{\mathcal{F}}_t\right]. \qquad \square$$

We are now ready to give the proof of the verification theorem.

Proof of Theorem 2.3.1. By assumption $\theta u 1_{[0,t]} \in \mathcal{H}^1$ for any $t \in (0,T)$, an application of Proposition A.1.1 with $G = \theta u_\tau$ for any $\tau < T$ yields $\theta u \in \cap_{\tau\in(0,T)} \mathcal{L}_{\mathcal{F}}^{2,\infty}(0,\tau)$.

The stochastic HJB equation associated with our optimization problem is given by the following BSPDE:

$$\begin{cases} -dV_t(x,y) = \Big[\mathcal{L}V_t(x,y) + \mathcal{M}\Psi_t(x,y) + \operatorname*{ess\,inf}_{\xi,\rho}\Big\{ -\xi D_x V_t(x,y) + \eta_t(y)|\xi|^2 \\ \qquad + \lambda_t(y)|x|^2 + \int_{\mathcal{Z}} \left\{V_t(x-\rho,y) - V_t(x,y) + \gamma_t(y,z)|\rho|^2\right\}\mu(dz)\Big\}\Big]\,dt \\ \qquad - \Psi_t(x,y)\,dW_t, \qquad (t,x,y) \in [0,T)\times\mathbb{R}\times\mathbb{R}^d; \\ V_T(x,y) = +\infty\cdot 1_{x\neq 0}, \qquad (x,y)\in\mathbb{R}\times\mathbb{R}^d. \end{cases}$$

It is easy to show that the pair $V_t(x,y) := u_t(x)|x|^2$ and $\Psi_t(x,y) := \psi_t(y)|x|^2$ solves the above equation if and only if (u,ψ) solves (2.4). This shows that (ξ^*,ρ^*) is the candidate optimal strategy. It therefore remains to show that (ξ^*,ρ^*) is admissible and attains the minimal cost.

In order to show admissibility, we plug the explicit expression for (ξ^*,ρ^*) into the state process and get

$$x_t^* := x\prod_{0<s\le t}\left\{1 - \int_Z \frac{u_s(y_s^{0,y})}{\gamma_s(y_s^{0,y},z) + u_s(y_s^{0,y})}\,\pi(dz,\{s\})\right\}\exp\left(-\int_0^t \frac{u_s(y_s^{0,y})}{\eta_s(y_s^{0,y})}\,ds\right)$$

for $t\in[0,t)$. Hence,

$$\begin{aligned} |x_t^*| &\le |x|\exp\left(-\int_0^t \frac{u_s(y_s^{0,y})}{\eta_s(y_s^{0,y})}\,ds\right) \\ &\le |x|\exp\left(-\int_0^t \frac{c_0}{\Lambda(T-s)}\,ds\right) = |x|\left(\frac{T-t}{T}\right)^{c_0/\Lambda} \xrightarrow{t\uparrow T} 0. \end{aligned}$$

From the definition of (ξ^*,ρ^*), we immediately infer that $\rho^*\in\mathcal{L}^2_{\bar{\mathcal{F}}}(0,T;L^2(\mathcal{Z}))$ and $\xi^*\in\mathcal{L}^2_{\bar{\mathcal{F}}}(0,t;\mathbb{R})$ for any $t\in(0,T)$. Moreover, the associated state sequence x^* is monotone.

In order to show that (ξ^*,ρ^*) is admissible and that the cost functional attains its minimum at (ξ^*,ρ^*), we notice that the process $\theta(y_t^{0,y})u_t(y_t^{0,y})$ satisfies the assumptions of Lemma 2.3.2 so we can apply the generalized Itô-Kunita-Wentzell formula. A subsequent application of the standard Itô formula to the product of θ^{-1} and θu yields the stochastic differential equation for $u_t(y_t^{0,y})$.

Applying the standard Itô formula again, this time to $u_t(y_t^{0,y})|x_t^{0,x;\xi,\rho}|^2$, we finally obtain the SDE for the candidate value function. A tedious but straightforward computation shows that for all admissible strategies (ξ,ρ) it holds for almost every $y\in\mathbb{R}^d$ that

$$\begin{aligned}
&u_t(y)|x|^2 - E\left[u_\tau(y_\tau^{t,y})\big|x_\tau^{t,x;\xi,\rho}\big|^2\Big|\bar{\mathcal{F}}_t\right]\\
&= E\Bigg[\int_t^\tau \Big\{2u_s(y_s^{t,y})x_s^{t,x;\xi,\rho}\xi_s + u_s(y_s^{t,y})\int_{\mathcal{Z}}\left\{2\rho_s(z)x_s^{t,x;\xi,\rho} - |\rho_s(z)|^2\right\}\mu(dz)\\
&+ \lambda_s(y_s^{t,y})|x_s^{t,x;\xi,\rho}|^2 - \int_{\mathcal{Z}}\frac{|u_s(y_s^{t,y})|^2|x_s^{t,x;\xi,\rho}|^2}{\gamma_s(y_s^{t,y},z)+u_s(y_s^{t,y})}\mu(dz) - \frac{|u_s(y_s^{t,y})|^2|x_s^{t,x;\xi,\rho}|^2}{\eta_s(y_s^{t,y})}\Big\}ds\Bigg|\bar{\mathcal{F}}_t\Bigg]\\
&\le E\Bigg[\int_t^\tau\Big\{\eta_s(y_s^{t,y})|\xi_s|^2 + \lambda_s(y_s^{t,y})\big|x_s^{t,x;\xi,\rho}\big|^2 + \int_{\mathcal{Z}}\gamma_s(y_s^{t,y},z)|\rho_s(z)|^2\,\mu(dz)\Big\}\,ds\Bigg|\bar{\mathcal{F}}_t\Bigg]
\end{aligned}\tag{2.13}$$

for all $0\le t\le\tau<T$. In view of Lemma 2.3.4 we may with no loss of generality assume that process $x^{0,x;\xi,\rho}$ is monotone and hence,

$$\lim_{\tau\to T}E\left[u_\tau(y_\tau^{t,y})|x_\tau^{t,x;\xi,\rho}|^2\Big|\bar{\mathcal{F}}_t\right]\le\lim_{\tau\to T}\frac{c_1}{T-\tau}C(T-\tau)E\left[\int_\tau^T|\xi_s|^2\,ds\Bigg|\bar{\mathcal{F}}_t\right]=0.$$

Thus, taking the limit $\tau\to T$ in (2.13) yields that $J_t(x,y;\xi,\rho)\le u_t(y)x^2$ for any admissible control (ξ,ρ). For (ξ^*,ρ^*) we have equality in (2.13), which implies $u_t(y)x^2 = J_t(x,y;\xi^*,\rho^*)$. But this in particular shows $\xi^*\in L^2_{\bar{\mathcal{F}}}(0,T;\mathbb{R})$, thus (ξ^*,ρ^*) is admissible, attains the minimal cost, and hence is optimal. □

2.4. Existence of a solution to BSPDE (2.4)

As a result of the verification theorem there exists at most one solution (u,ψ) to (2.4) that satisfies (2.10) and (2.11). In this section, we prove existence of a solution with these properties. To this end, we set

$$\hat{F}(t,y,\phi(y)) := F(t,y,|\phi(y)|),\quad (t,y,\phi)\in\mathbb{R}_+\times\mathbb{R}^d\times L^0(\mathbb{R}^d),\tag{2.14}$$

and construct the solution as the limit of a sequence of such a solution to a family of BSPDEs with driver $\hat{F}$ and finite increasing terminal values. More precisely, for each $N\in\mathbb{N}$, we consider the BSPDE

$$\begin{cases}-dv_t^N(y) = \{\tilde{\mathcal{L}}v_t^N(y)+\tilde{\mathcal{M}}\zeta_t^N(y)+\theta(y)\hat{F}(t,y,(\theta^{-1}v_t^N)(y))\}\,dt\\ \qquad\qquad - \zeta_t^N(y)\,dW_t,\qquad (t,y)\in[0,T]\times\mathbb{R}^d;\\ v_T^N(y) = N\theta(y),\qquad y\in\mathbb{R},\end{cases}\tag{2.15}$$

that corresponds to the singular BSPDE (2.6), with the pair (F,∞) being replaced by $(\hat{F},N\theta)$. We cannot appeal directly to Proposition A.1.1 to prove existence of a solution to the preceding BSPDE, due to the quadratic dependence of the driver $\hat{F}$ on $|\phi(y)|$ in (2.14). However, we expect v^N to be finite and hence to be able to construct a solution by a standard truncation argument.

Proposition 2.4.1. *Let assumptions* $(\mathcal{A}1)-(\mathcal{A}3)$ *be satisfied. For each* $N \in \mathbb{N}$, *there exists a unique solution to* (2.15) *such that*

$$(v^N, \zeta^N) \in (\mathcal{H}^1 \cap \mathcal{L}^{2,\infty}_{\mathcal{F}}(0,T)) \times \mathcal{L}^2_{\mathcal{F}}(0,T; H^{1,2}(\mathbb{R}^d))$$

and $\theta^{-1} v^N \in \mathcal{L}^\infty_{\mathcal{F}}(0,T; L^\infty(\mathbb{R}^d))$.

Proof. For each $M \in \mathbb{N}$ there exists a unique solution

$$(v^{N,M}, \zeta^{N,M}) \in (\mathcal{H}^1 \cap \mathcal{L}^{2,\infty}_{\mathcal{F}}(0,T)) \times \mathcal{L}^2_{\mathcal{F}}(0,T; H^{1,2}(\mathbb{R}^d))$$

to the BSPDE

$$\begin{cases} -dv^{N,M}_t(y) = \Big(\tilde{\mathcal{L}} v^{N,M}_t + \tilde{\mathcal{M}} \zeta^{N,M}_t + \theta\lambda - \displaystyle\int_{\mathcal{Z}} \frac{\theta^{-1}|v^{N,M}_t|^2}{\gamma_t(\cdot,z) + |\theta^{-1} v^{N,M}_t|}\, \mu(dz) \\ \qquad\qquad - \dfrac{(M \wedge |\theta^{-1} v^{N,M}_t|)|v^{N,M}_t|}{\eta_t} \Big)(y)\, dt - \zeta^{N,M}(y)\, dW_t, \ \ (t,y) \in [0,T] \times \mathbb{R}^d; \\ v^{N,M}_T(y) = N\theta(y), \quad y \in \mathbb{R}^d, \end{cases} \tag{2.16}$$

due to Proposition A.1.1. Putting

$$\hat{v}_t(y) = \theta(y)\left(N + \Lambda(T-t)\right),$$

we verify that $(\hat{v}, 0)$ is a solution of the above BSPDE with (λ, γ, M) being replaced by $(\Lambda, +\infty, 0)$. The comparison principle stated in Corollary A.1.2 yields

$$0 \le v^{N,M}_t(y) \le \hat{v}_t(y), \quad \mathbb{P} \otimes dt \otimes dy\text{-a.e.},$$

which implies for any $M \in \mathbb{N}$ that

$$0 \le \theta^{-1}(y) v^{N,M}_t(y) \le N + \Lambda T, \quad \mathbb{P} \otimes dt \otimes dy\text{-a.e.}$$

Hence, if $M > N + \Lambda T$, then $(v^{N,M}, \zeta^{N,M})$ does not depend on M and is in fact a solution to (2.15). This also yields uniqueness of solutions as (2.16) admits a unique solution for each $M \in \mathbb{N}$. □

The proof of Proposition 2.4.1 shows that the solution (v^N, ζ^N) to (2.15) coincides with that of (2.16) for some $M \in \mathbb{N}$. Hence, as an immediate consequence of Corollary A.1.2 we obtain the following comparison principle.

Corollary 2.4.2. *Let assumptions* $(\mathcal{A}1)-(\mathcal{A}3)$ *be satisfied and let* $(\bar{\lambda}, \bar{\gamma}, \bar{\eta})$ *satisfy the same conditions as* (λ, γ, η). *Suppose further that*

$$(\bar{v}, \bar{\zeta}) \in \mathcal{H}^1 \times \mathcal{L}^2_{\mathcal{F}}(0,T; H^{1,2}(\mathbb{R}^d))$$

with $\theta^{-1}\bar{v} \in \mathcal{L}^\infty_{\mathcal{F}}(0,T; L^\infty(\mathbb{R}^d))$, *is a solution to the following BSPDE:*

$$\begin{cases} -d\bar{v}_t(y) = \{\tilde{\mathcal{L}} \bar{v}_t(y) + \tilde{\mathcal{M}} \zeta^N_t(y) + \theta(y)\hat{F}(t,y,(\theta^{-1} v^N_t)(y))\}\, dt \\ \qquad\qquad - \bar{\zeta}_t(y)\, dW_t, \qquad (t,y) \in [0,T] \times \mathbb{R}^d; \\ \bar{v}_T(y) = G(y), \qquad y \in \mathbb{R}^d. \end{cases}$$

If $(G, \bar{\lambda}, \bar{\gamma}, \bar{\eta}) \geq (N, \lambda, \gamma, \eta)$, *respectively,* $(G, \bar{\lambda}, \bar{\gamma}, \bar{\eta}) \leq (N, \lambda, \gamma, \eta)$, *then for almost all* (ω, y) *it holds that*

$$\bar{v}_t(y) \geq v_t^N(y), \quad \textit{respectively,} \quad \bar{v}_t(y) \leq v_t^N(y), \quad \forall t \in [0, T].$$

We are now ready to prove existence of a solution to our singular BSPDE that satisfies the assumptions of the verification theorem.

Theorem 2.4.3. *Let assumptions* $(\mathcal{A}1)-(\mathcal{A}3)$ *be satisfied. Then the BSPDE* (2.4) *admits a solution* (u, ψ) *satisfying* (2.10) *and* (2.11).

Proof. By Proposition 2.4.1, for each $N > 2\Lambda + \kappa_0 \mu(\mathcal{Z})$, there exists a unique solution (v^N, ζ^N) to (2.15) such that $(v^N, \zeta^N) \in (\mathcal{H}^1 \cap \mathcal{L}^{2,\infty}_{\mathcal{F}}(0,T)) \times \mathcal{L}^2_{\mathcal{F}}(0,T; H^{1,2}(\mathbb{R}^d))$ and $\theta^{-1} v^N \in \mathcal{L}^\infty_{\mathcal{F}}(0,T; L^\infty(\mathbb{R}^d))$. If one replaces the triple (λ, γ, η) by $(\Lambda, +\infty, \Lambda)$ and $(0, 0, \kappa_0)$, respectively, then a direct computation shows that respective solutions to (2.15) are given by $(\bar{u}^N, 0)$ and $(\tilde{u}^N, 0)$, where

$$\begin{aligned}
\bar{u}_t^N(y) &:= \frac{\kappa_0 \mu(\mathcal{Z}) \theta(y)}{1 - \frac{N}{N + \kappa_0 \mu(\mathcal{Z})} e^{-\mu(\mathcal{Z})(T-t)}} - \kappa_0 \mu(\mathcal{Z}) \theta(y), \\
\tilde{u}_t^N(y) &:= \frac{2\Lambda \theta(y)}{1 - \frac{N - \Lambda}{N + \Lambda} \cdot e^{-2(T-t)}} - \Lambda \theta(y).
\end{aligned}$$

From Corollary 2.4.2, we conclude that for almost every $y \in \mathbb{R}^d$, it holds almost surely that

$$\bar{u}_t^N(y) \leq v_t^N(y) \leq \tilde{u}_t^N(y), \quad t \in [0, T).$$

Denoting by v the limit of the increasing sequence $\{v^N\}$, we deduce that for almost every $y \in \mathbb{R}^d$ that almost surely

$$\frac{\kappa_0 e^{-\mu(\mathcal{Z})T} \theta(y)}{T - t} \leq v_t(y) \leq \frac{\Lambda e^{2T} \theta(y)}{T - t}, \quad t \in [0, T). \tag{2.17}$$

Further, by dominated convergence,

$$\lim_{N \to \infty} \|\theta(\cdot) \hat{F}(\cdot, \cdot, (\theta^{-1} v_\cdot^N)(\cdot)) - \theta(\cdot) F(\cdot, \cdot, (\theta^{-1} v_\cdot)(\cdot))\|_{\mathcal{L}^2_{\mathcal{F}}(0,\tau; L^2(\mathbb{R}^d))} = 0, \quad \tau \in (0, T).$$

We now use v to construct the desired solution by analyzing a BSPDE on $[0, \tau]$ with terminal value v_τ. More precisely, let us denote by

$$(\bar{v}, \zeta) \in \left(\mathcal{L}^2_{\mathcal{F}}(0, \tau; H^{1,2}(\mathbb{R}^d)) \cap \mathcal{S}^2_{\mathcal{F}}([0, \tau]; L^2(\mathbb{R}^d))\right) \times \mathcal{L}^2_{\mathcal{F}}(0, \tau; L^2(\mathbb{R}^d))$$

the unique solution for the following BSPDE (guaranteed by Proposition A.1.1 as $v_\tau \in \mathcal{L}^2(\Omega, \mathcal{F}_\tau; L^2(\mathbb{R}^d))$ by (2.17)):

$$\begin{cases}
-d\bar{v}_t(y) = \{\tilde{\mathcal{L}} \bar{v}_t(y) + \tilde{\mathcal{M}} \zeta_t(y) + \theta(y) \hat{F}(t, y, (\theta^{-1} v_t)(y))\} \, dt \\
\qquad\qquad - \zeta_t(y) \, dW_t, \qquad (t, y) \in [0, \tau) \times \mathbb{R}^d; \\
\bar{v}_\tau(y) = v_\tau(y), \qquad y \in \mathbb{R}^d.
\end{cases}$$

We use this equation to show that v lies in the right space. In view of estimate (A.3) in Proposition A.1.1, we have as $N \to +\infty$,

$$\begin{aligned}
&\|(v^N - \bar{v})1_{[0,\tau]}\|_{\mathcal{H}^0} + \|\zeta^N - \zeta\|_{\mathcal{L}^2_{\mathcal{F}}(0,T;L^2(\mathbb{R}^d))} \\
&\qquad \leq C\Big(\|v^N_\tau - v_\tau\|_{L^2(\Omega,\mathcal{F}_\tau;L^2(\mathbb{R}^d))} + \|\theta\hat{F}(\cdot,\cdot,(\theta^{-1}v^N_\cdot)(\cdot)) \\
&\qquad\qquad - \theta F(\cdot,\cdot,(\theta^{-1}v_\cdot)(\cdot))\|_{\mathcal{L}^2_{\mathcal{F}}(0,\tau;L^2(\mathbb{R}^d))}\Big) \longrightarrow 0.
\end{aligned}$$

Thus,

$$\bar{v} = v1_{[0,\tau]} \in \mathcal{H}^0 = \mathcal{L}^2_{\mathcal{F}}(0,\tau;H^{1,2}(\mathbb{R}^d)) \cap \mathcal{S}^2_{\mathcal{F}}([0,\tau];L^2(\mathbb{R}^d)).$$

Hence, for each $\delta \in (0,\tau)$ there is $\tilde{\tau} \in (\tau-\delta,\tau]$ such that $v_{\tilde{\tau}} \in L^2(\Omega,\mathcal{F}_{\tilde{\tau}};H^{1,2}(\mathbb{R}^d))$, and by Proposition A.1.1, we further have

$$(v1_{[0,\tilde{\tau}]},\zeta 1_{[0,\tilde{\tau}]}) \in (\mathcal{L}^{2;\infty}_{\mathcal{F}}(0,\tilde{\tau}) \cap \mathcal{H}^1) \times \mathcal{L}^2_{\mathcal{F}}(0,\tilde{\tau};H^{1,2}(\mathbb{R}^d)).$$

This shows that $(u,\psi) := (\theta^{-1}v,\theta^{-1}\zeta)$ is a solution to BSPDE (2.4) with the desired properties. □

2.5. Uniqueness and regularity

In this section we show that the solution to the BSPDE (2.4) constructed in the previous section is the unique non-negative solution to (2.4). Subsequently, using the existing L^p-theory of BSPDEs, we consider the regularity of the solution.

2.5.1. Uniqueness

The following uniqueness result is based on the observation that any non-negative solution to (2.4) automatically satisfies the growth condition of the verification theorem.

Theorem 2.5.1. *Under assumptions $(\mathcal{A}1)-(\mathcal{A}3)$, the solution (u,ψ) given in Theorem 2.4.3 is the unique non-negative solution to (2.4) in the sense that if $(\bar{u},\bar{\psi})$ is another solution that satisfies (2.10) and $\bar{u} \geq 0$, $\mathbb{P}\otimes dt\otimes dx$-a.e., then*

$$\bar{u}_t(y) = u_t(y), \quad \mathbb{P}\otimes dt\otimes dy\text{-a.e.}$$

Before proving Theorem 2.5.1, we state the following lemma along with a sketched proof.

Lemma 2.5.2. *Let ϑ be a continuous and bounded function satisfying assumption $(\mathcal{A}1)$ and the Lipschitz continuity of assumption $(\mathcal{A}2)$. Then, for any $(u,\xi) \in H^{2,2}(\mathbb{R}^d)\times H^{1,2}(\mathbb{R}^d)$, we have for $i,j,l = 1,\dots,d$:*

(i)
$$\begin{aligned}
\|\vartheta\partial_{y_i}\xi\|_{L^2(\mathbb{R}^d)} + \|\vartheta\xi\|_{L^2(\mathbb{R}^d)} &\leq \|\vartheta\|_{L^\infty(\mathbb{R}^d)}\left(\|\partial_{y_i}\xi\|_{L^2(\mathbb{R}^d)} + \|\xi\|_{L^2(\mathbb{R}^d)}\right); \\
\|\vartheta\partial^2_{y_iy_j}u\|_{L^2(\mathbb{R}^d)} + \|\vartheta\partial_{y_l}u\|_{L^2(\mathbb{R}^d)} &\leq \|\vartheta\|_{L^\infty(\mathbb{R}^d)}\left(\|\partial^2_{y_iy_j}u\|_{L^2(\mathbb{R}^d)} + \|\partial_{y_l}u\|_{L^2(\mathbb{R}^d)}\right).
\end{aligned}$$

$$(ii)\qquad \begin{aligned} &|\langle u,\, \vartheta\partial^2_{y_iy_j}u\rangle + \langle \partial_{y_i}u,\, \vartheta\partial_{y_j}u\rangle| \le L\,\langle |u|,\, |\partial_{y_j}u|; \\ &\qquad |\langle u,\, \vartheta\partial_{y_i}\xi\rangle + \langle \partial_{y_i}u,\, \vartheta\xi\rangle| \le L\,\langle |u|,\, |\xi|\rangle. \end{aligned}$$

Sketch of the proof. The assertion (i) can be verified via the well-known Hölder inequality. Let $\zeta \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative function such that $\int_{\mathbb{R}^d}\zeta(y)\,dy = 1$ and define $\zeta_k(y) = k^d\zeta(ky)$, $k = 1, 2, \dots$, and

$$\vartheta_k(y) = \int_{\mathbb{R}^d} \vartheta(y-x)\zeta_k(x)\,dx.$$

Then we have

$$\lim_{k\to\infty} \|\vartheta_k - \vartheta\|_{L^\infty(\mathbb{R}^d)} = 0, \tag{2.18}$$

and for each k, ϑ_k satisfies $(\mathcal{A}1)$ and $(\mathcal{A}2)$ with the same parameters Λ and L. Moreover, for each k,

$$\begin{aligned} &|\langle u,\, \vartheta_k\partial^2_{y_iy_j}u\rangle + \langle \partial_{y_i}u,\, \vartheta_k\partial_{y_j}u\rangle| \le L\langle |u|,\, |\partial_{y_j}u|\rangle, \quad i,j = 1,\dots,d; \\ &\qquad |\langle u,\, \vartheta_k\partial_{y_i}\xi\rangle + \langle \partial_{y_i}u,\, \vartheta_k\xi\rangle| \le L\langle |u|,\, |\xi|\rangle, \quad i = 1,\dots,d. \end{aligned}$$

Hence, in view of (2.18) and (i), the assertion (ii) follows from letting $k \to \infty$. □

Proof of Theorem 2.5.1. In view of Theorem 2.3.1, to establish the uniqueness statement it is sufficient to verify that $\bar{u}$ satisfies the growth condition (2.11).

Set $(\bar{v}, \bar{\zeta}) = (\theta\bar{u}, \theta\bar{\psi})$ and for $N \in \mathbb{N}$, let (v^N, ζ^N) be the unique solution to (2.15). From the proof for Theorem 2.4.3 we see that to establish the lower bound in (2.11) one needs only to prove

$$\bar{v}_t(y) \ge v_t^N(y), \quad \mathbb{P}\otimes dt\otimes dy\text{-a.e.}$$

Putting $(\tilde{v}, \tilde{\zeta}) = (v^N - \bar{v}, \zeta^N - \bar{\zeta})$ and noticing that for the moment one only has that $\eta^{-1}|\bar{v}|^2$ lies in $\mathcal{L}^1_{\mathcal{F}}(0,t;L^1(\mathbb{R}^d))$ instead of $\mathcal{L}^2_{\mathcal{F}}(0,t;L^2(\mathbb{R}^d))$, we apply the inequality for BSPDEs stated in Lemma A.1.3 in the appendix. Since

$$\left(F(t,y,(\theta^{-1}\phi_1)(y)) - F(t,y,(\theta^{-1}\phi_2)(y))\right)(\phi_1-\phi_2)^+(y) \le 0, \quad \mathbb{P}\otimes dt\otimes dy\text{-a.e.},$$

for any pair of non-negative measurable functions ϕ_1 and ϕ_2 on $\mathbb{R}^d$, and because σ and $\bar{\sigma}$ are bounded and Lipschitz continuous, we obtain from that lemma for $0 < t < \tau < T$,

$$\begin{aligned} &E\left[\|\tilde{v}_t^+\|^2_{L^2(\mathbb{R}^d)} + \int_t^\tau \|\tilde{\zeta}_s 1_{\tilde{v}>0}\|^2_{L^2(\mathbb{R}^d)}\,ds\right] \\ &\le E\left[\|\tilde{v}_\tau^+\|^2_{L^2(\mathbb{R}^d)} + \int_t^\tau 2\langle \tilde{v}_s^+,\, a_s^{ij}\partial^2_{y_iy_j}\tilde{v}_s + \sigma_s^{jr}\partial_{y_j}\tilde{\zeta}_s^r + \tilde{b}_s^i\partial_{y_i}\tilde{v}_s + \beta_s^{\mathcal{T}}\tilde{\zeta}_s + c_s\tilde{v}_s\rangle\,ds\right], \end{aligned}$$

where the summation convention is applied. This yields by Lemma 2.5.2,

$$
\begin{aligned}
&E\left[\|\tilde v_t^+\|^2_{L^2(\mathbb{R}^d)}+\int_t^\tau \|\tilde\zeta_s 1_{\tilde v>0}\|^2_{L^2(\mathbb{R}^d)}\,ds\right]\\
&\le E\Bigg[\|\tilde v_\tau^+\|^2_{L^2(\mathbb{R}^d)}+\int_t^\tau \Big\{C(L,d)\langle \tilde v_s^+,\,|D\tilde v_s^+|+|\tilde\zeta_s 1_{\tilde v>0}|\rangle\\
&\qquad\qquad +2\langle \tilde v_s^+,\,\tilde b_s^i\partial_{y_i}\tilde v_s+\beta_s^{\mathcal{T}}\tilde\zeta_s+c_s\tilde v_s\rangle-2\langle\partial_{y_j}\tilde v_s^+,\,a_s^{ij}\partial_{y_i}\tilde v_s^++\sigma_s^{jr}\tilde\zeta_s^r\rangle\Big\}\,ds\Bigg].
\end{aligned}
$$

Applying the Hölder inequality twice, we obtain for $\varepsilon,\delta>0$,

$$
\begin{aligned}
&E\left[\|\tilde v_t^+\|^2_{L^2(\mathbb{R}^d)}+\int_t^\tau \|\tilde\zeta_s 1_{\tilde v>0}\|^2_{L^2(\mathbb{R}^d)}\,ds\right]\\
&\le E\Bigg[\|\tilde v_\tau^+\|^2_{L^2(\mathbb{R}^d)}+\int_t^\tau \Big\{C(\varepsilon,\kappa,d,L)\|\tilde v_s^+\|^2_{L^2(\mathbb{R}^d)}+\frac{\kappa}{4}\|D\tilde v_s^+\|^2_{L^2(\mathbb{R}^d)}\\
&\qquad +\varepsilon\|\tilde\zeta_s 1_{\tilde v>0}\|^2_{L^2(\mathbb{R}^d)}-2\langle\partial_{y_j}\tilde v_s^+,\,a_s^{ij}\partial_{y_i}\tilde v_s^++\sigma_s^{jr}\tilde\zeta_s^r\rangle\Big\}\,ds\Bigg]\\
&\le E\Bigg[\|\tilde v_\tau^+\|^2_{L^2(\mathbb{R}^d)}+\int_t^\tau \Big\{C(\delta,\varepsilon,\kappa,d,L)\|\tilde v_s^+\|^2_{L^2(\mathbb{R}^d)}+\frac{\kappa}{4}\|D\tilde v_s^+\|^2_{L^2(\mathbb{R}^d)}\\
&\quad +\left(\tfrac{1}{1+\delta}+\varepsilon\right)\|\tilde\zeta_s 1_{\tilde v>0}\|^2_{L^2(\mathbb{R}^d)}-\langle\partial_{y_i}\tilde v^+(s),\,\left(2a_s^{ij}-(1+\delta)\sigma_s^{jr}\sigma_s^{ir}\right)\partial_{y_j}\tilde v_s^+\rangle\Big\}\,ds\Bigg]\\
&= E\Bigg[\|\tilde v_\tau^+\|^2_{L^2(\mathbb{R}^d)}+\int_t^\tau \Big\{C\|\tilde v_s^+\|^2_{L^2(\mathbb{R}^d)}+\frac{\kappa}{4}\|D\tilde v_s^+\|^2_{L^2(\mathbb{R}^d)}\\
&\quad +\left(\tfrac{1}{1+\delta}+\varepsilon\right)\|\tilde\zeta_s 1_{\tilde v>0}\|^2_{L^2(\mathbb{R}^d)}-\langle\partial_{y_i}\tilde v^+(s),\,\left(\bar\sigma_s^{jr}\bar\sigma_s^{ir}-\delta\sigma_s^{jr}\sigma_s^{ir}\right)\partial_{y_j}\tilde v_s^+\rangle\Big\}\,ds\Bigg].
\end{aligned}
$$

By Assumption $(\mathcal{A}1)$ and $(\mathcal{A}3)$, choosing δ small such that $\delta\Lambda^2\le\kappa/4$, and and choosing moreover ε small such that $\frac{1}{\delta+1}+\varepsilon\le 1$, yields

$$
\begin{aligned}
&E\left[\|\tilde v_t^+\|^2_{L^2(\mathbb{R}^d)}+\int_t^\tau \|\tilde\zeta_s 1_{\tilde v>0}\|^2_{L^2(\mathbb{R}^d)}\,ds\right]\\
&\le E\left[\|\tilde v_\tau^+\|^2_{L^2(\mathbb{R}^d)}+\int_t^\tau \left\{C\|\tilde v_s^+\|^2_{L^2(\mathbb{R}^d)}+\|\tilde\zeta_s 1_{\tilde v>0}\|^2_{L^2(\mathbb{R}^d)}-\frac{\kappa}{2}\|D\tilde v_s^+\|^2_{L^2(\mathbb{R}^d)}\right\}ds\right].
\end{aligned}
$$

By Gronwall's inequality this implies

$$
E\left[\|\tilde v_t^+\|^2_{L^2(\mathbb{R}^d)}\right]\le CE\left[\|\tilde v_\tau^+\|^2_{L^2(\mathbb{R}^d)}\right],
$$

where C is independent of τ and t. As $\theta^{-1}v^N\in\mathcal{L}^\infty_{\mathscr{F}}(0,T;L^\infty(\mathbb{R}^d))$ and $v^N\in\mathcal{H}^1$ by Proposition 2.4.1, and

$$
\tilde v^+=(v^N-\bar v)^+\le|v^N|,\quad \mathbb{P}\otimes dt\otimes dy\text{-a.e.},
$$

we have by Fatou's lemma

$$\int_{[0,T]\times\mathbb{R}^d} E[|\tilde{v}_t^+(y)|^2]\,dydt \le CT\limsup_{\tau\uparrow T}\int_{\mathbb{R}^d} E[|\tilde{v}_\tau^+(y)|^2]\,dy$$
$$\le CT\int_{\mathbb{R}^d} E\left[\limsup_{\tau\uparrow T}|\tilde{v}_\tau^+(y)|^2\right]dy = 0.$$

Hence, the lower bound of (2.11) holds for $\overline{u}$.

To establish the upper bound in (2.11) we extend an argument given in [GHS16] and consider the deterministic function

$$\hat{u}_t := \Lambda\coth(T-t) = \frac{2\Lambda}{1-e^{-2(T-t)}} - \Lambda \le \frac{\Lambda e^{2T}}{T-t}.$$

Then, $(\hat{u},0)$ is a solution to (2.4) with (λ,γ,η) being replaced by $(\Lambda,+\infty,\Lambda)$. Moreover, $(\hat{u},0)$ remains a solution when shifted in time, i.e., for $\delta\in[0,T)$ the pair $(\hat{u}_{\cdot+\delta},0)$ is the solution to (2.4) associated with $(\Lambda,+\infty,\Lambda)$, but with a singularity at $t=T-\delta$. Hence, noting that

$$\left(F(t,y,(\theta^{-1}\phi_1)(y)) - \Lambda + \Lambda^{-1}|(\theta^{-1}\phi_2)(y)|^2\right)(\phi_1-\phi_2)^+(y)\le 0, \quad \mathbb{P}\otimes dt\otimes dy\text{-a.e.},$$

for any pair of non-negative measurable functions ϕ_1 and ϕ_2 on $\mathbb{R}^d$, using arguments similar to those used in the first part of this proof, we conclude

$$\int_{[0,T-\delta]\times\mathbb{R}^d} E[|(\theta\bar{u}_t-\theta\hat{u}_{t+\delta})^+(y)|^2]\,dy\,dt$$
$$\le C(T-\delta)\int_{\mathbb{R}^d} E\left[\limsup_{\tau\uparrow T-\delta}|(\theta\bar{u}_\tau-\theta\hat{u}_{\tau+\delta})^+(y)|^2\right]dy = 0.$$

This yields,

$$\bar{u}_t(y)\le\frac{\Lambda e^{2T}}{T-\delta-t}, \quad \mathbb{P}\otimes dt\otimes dy\text{-a.e.}$$

Finally, letting $\delta\to 0$ we obtain the desired upper bound. □

2.5.2. Regularity

We proved so far that, under assumptions $(\mathcal{A}1)-(\mathcal{A}3)$, the BSPDE (2.4) admits a unique non-negative solution (u,ψ) that satisfies (2.10). This solution automatically satisfies the growth condition (2.11) and $V(t,y,x) := u_t(y)x^2$ coincides with the value function of (2.2) for almost every $y\in\mathbb{R}^d$.

Inspired by the L^p-theory $(p>2)$ of BSPDEs, we now prove additional regularity properties of u under the following additional assumption:

$(\mathcal{A}4)$ σ is spatially invariant (does not depend on y).

Theorem 2.5.3. *Under assumptions* $(\mathcal{A}1)-(\mathcal{A}4)$*, let* (u,ψ) *be the unique nonnegative solution to* (2.4) *that satisfies* (2.10)*. Then, for any* $p\in(2,+\infty)$*,*

$$\theta(\cdot)u_{\cdot}\Big(\,\cdot+\int_0^{\cdot}\sigma_s\,dW_s\Big)\in\bigcap_{\tau\in(0,T)}\bigcap_{\delta\in(0,1)}\mathcal{L}^{2,\infty}_{\mathcal{F}}(0,\tau)\cap\mathcal{S}^p_{\mathcal{F}}([0,\tau];C^{\delta}(\mathbb{R}^d)).$$

Furthermore, the function $V(t,y,x):=u_t(y)x^2$ *coincides with the value function of* (2.2) *for every* $y\in\mathbb{R}^d$*.*

Proof. For each $N\in\mathbb{N}$, let (v^N,ζ^N) be the unique solution to the BSPDE (2.15). Our goal is to derive additional regularity properties under $(\mathcal{A}4)$ using the L^p-theory for BSPDEs developed in [DQT12].

The results of [DQT12] do not allow the linear term $\beta^{\mathcal{T}}\zeta^N$ in the drift part of the BSPDE, though. To overcome this problem, we make the following change of variables:

$$\begin{aligned}
y^y_t &:= y+\int_0^t\sigma_s\,dW_s,\quad (t,y)\in[0,T]\times\mathbb{R}^d;\\
\bar a_s(y) &:= \frac12\bar\sigma_s(y)\bar\sigma_s^{\mathcal{T}}(y),\quad y\in\mathbb{R}^d;\\
(\tilde u^N_t,\tilde\psi^N_t)(y) &:= (\theta^{-1}v^N_t,\theta^{-1}\zeta^N_t+\sigma_t^{\mathcal{T}}D(\theta^{-1}v^N_t))(y^y_t),\quad (t,y)\in[0,T]\times\mathbb{R}^d;\\
(\tilde v^N_t,\tilde\zeta^N_t)(y) &:= (\theta\tilde u^N_t,\theta\tilde\psi^N_t)(y),\quad (t,y)\in[0,T]\times\mathbb{R}^d.
\end{aligned}$$

Then, applying the Itô-Wentzell formula [Kry12, Theorem 1] for distribution-valued processes, we have almost surely that

$$\begin{aligned}
\tilde v^N_t(y) &= N\theta(y)+\int_t^T\Big\{\operatorname{tr}\left(\bar a_s(y^y_s)D^2\tilde v^N_s(y)\right)+\bar b^{\mathcal{T}}_s(y)D\tilde v^N_s(y)+\bar c_s(y)\tilde v^N_s(y)\\
&\quad+\theta(y)\hat F(s,y^y_s,(\theta^{-1}\tilde v^N_s)(y))\Big\}\,ds-\int_t^T\tilde\zeta^N_s(y)\,dW_s,\quad dy\text{-a.e.}\quad\forall t\in[0,T]
\end{aligned}\tag{2.19}$$

with

$$\begin{aligned}
\bar b^i_t(y) &:= b^i_t(y^y_t)+\frac{4q}{1+|y|^2}\sum_{j=1}^d a^{ij}_t(y^y_t)y^j,\quad i=1,\dots,d;\\
\bar c_t(y) &:= \frac{2q}{1+|y|^2}\Big(\operatorname{tr}(a_t(y^y_t))+\sum_{i=1}^d y^ib^i_t(y^y_t)+\frac{2(q-1)}{1+|y|^2}\sum_{i,j=1}^d a^{ij}_t(y^y_t)y^iy^j\Big).
\end{aligned}$$

From this representation we see that we also have a BSDE representation of $(\tilde v^N,\tilde\zeta^N)$ from which we will obtain strong regularity properties. Specifically, by Proposition A.1.1, there exists a unique solution

$$(\bar v^N,\bar\zeta^N)\in\Big(\mathcal{H}^1\cap\mathcal{L}^{2,\infty}_{\mathcal{F}}(0,T)\Big)\times\mathcal{L}^2_{\mathcal{F}}(0,T;H^{1,2}(\mathbb{R}^d)),$$

to the BSPDE

$$\begin{cases} -d\bar{v}_t^N(y) = \Big\{ \operatorname{tr}\left(\bar{a}_t(y_t^y)D^2\bar{v}_t^N(y)\right) + \bar{b}_t^{\mathcal{T}}(y)D\bar{v}_t^N(y) + \bar{c}_t(y)\bar{v}_t^N(y) + \theta(y)\bar{\lambda}_t(y_t^y) \\ \qquad - \dfrac{\left|\tilde{v}_t^N(y)\bar{v}_t^N(y)\right|}{\theta(y)\bar{\eta}_t(y_t^y)} - \displaystyle\int_{\mathcal{Z}} \frac{\theta^{-1}(y)|\bar{v}_t^N(y)|^2}{\bar{\gamma}_t(y_t^y,z) + |\theta^{-1}(y)\bar{v}_t^N(y)|}\mu(dz)\Big\}\, dt \\ \qquad - \bar{\zeta}_t^N(y)\, dW_t, \qquad (t,y) \in [0,T]\times\mathbb{R}^d; \\ \bar{v}_T^N(y) = N\theta(y), \qquad y \in \mathbb{R}^d. \end{cases} \tag{2.20}$$

By definition, the solution satisfies (2.19). As $\theta^{-1}\tilde{v}^N \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty(\mathbb{R}^d))$ we can use the comparison principle stated in Corollary A.1.2 to deduce (similarly to the proof of Proposition 2.4.1) that $\theta^{-1}\bar{v}^N \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty(\mathbb{R}^d))$. Hence, by [DQT12, Proposition 6.4], we further have

$$\bar{v}^N \in \mathcal{S}^p_{\mathcal{F}}([0,T];H^{1,p}(\mathbb{R}^d)) \cap \mathcal{L}^p_{\mathcal{F}}(0,T;H^{2,p}(\mathbb{R}^d)) \quad \text{for any } p \in (2,+\infty).$$

Thus, by Sobolev embedding theorem, $\bar{v}^N \in \mathcal{S}^p_{\mathcal{F}}([0,T];C^\delta(\mathbb{R}^d))$, for any $\delta \in (0,1)$. Therefore, $\bar{v}_t^N(y)$ is almost surely continuous in $(t,y) \in [0,T]\times\mathbb{R}^d$.

Next, we are going to show that

$$\tilde{v}_t^N(y) = \bar{v}_t^N(y), \quad \mathbb{P}\otimes dy\text{-a.e.}$$

To this end, we show that both $(\tilde{v}^N,\tilde{\zeta}^N)$ and $(\bar{v}^N,\bar{\zeta}^N)$ satisfy the same BSDE. Specifically, let

$$\tilde{y}_t^{s,y} := y + \int_s^t \bar{b}_r(\tilde{y}_r^{s,y})\, dr + \int_s^t \bar{\sigma}_r(y_r^{\tilde{y}_r^{s,y}})\, dB_r, \quad 0 \le s \le t \le T.$$

Since $(\tilde{v}_t^N,\ \tilde{\zeta}_t^y)(y) = \theta(y)(\theta^{-1}v_t^N,\ \theta^{-1}\zeta_t^N + \sigma_t^{\mathcal{T}}D(\theta^{-1}v_t^N))(y_t^y)$, one checks through standard but tedious computations that both $\bar{v}^N$ and $\tilde{v}^N$ are bounded and satisfy the following BSDE:

$$\begin{aligned} \check{v}_t^N(\tilde{y}_t^{s,y}) = N\theta(\tilde{y}_T^{s,y}) + \int_t^T \Big\{ & \bar{c}_r(\tilde{y}_r^{s,y})\check{v}_r^N(\tilde{y}_r^{s,y}) + \theta(\tilde{y}_r^{s,y})\bar{\lambda}_t(y_r^{\tilde{y}_r^{s,y}}) \\ - \frac{\theta^{-1}(\tilde{y}_r^{s,y})\,|\check{v}_t(\tilde{y}_r^{s,y})\tilde{v}_t(\tilde{y}_r^{s,y})|}{\bar{\eta}_t(y_r^{\tilde{y}_r^{s,y}})} & - \int_{\mathcal{Z}} \frac{\theta^{-1}(\tilde{y}_r^{s,y})|\check{v}_t(\tilde{y}_r^{s,y})|^2}{\bar{\gamma}_t(y_r^{\tilde{y}_r^{s,y}},z) + \theta^{-1}(\tilde{y}_r^{s,y})|\check{v}_t(\tilde{y}_r^{s,y})|}\mu(dz)\Big\}\, dr \\ & - \int_t^T \check{\zeta}_r^N(\tilde{y}_r^{s,x})\, dW_r - \int_t^T \bar{\sigma}_r^{\mathcal{T}}(\tilde{y}_r^{s,y})D\check{v}_r^N(\tilde{y}_r^{s,x})\, dB_r. \end{aligned}$$

This BSDE has a unique solution. In view of Lemma 2.3.2 and Corollary 2.3.3, we therefore conclude

$$\tilde{v}_t^N(\tilde{y}_t^{s,y}) = \bar{v}_t^N(\tilde{y}_t^{s,y}), \quad \mathbb{P}\otimes dy\text{-a.e.} \quad \forall\, 0 \le s \le t \le T,$$

where we note that both $\tilde{v}^N$ and $\bar{v}^N$ belong to $\mathcal{H}^1 \cap \mathcal{L}^{2,\infty}_{\mathcal{F}}(0,T)$. Taking $s = t$, we have

$$\tilde{v}_t^N(y) = \bar{v}_t^N(y), \quad \mathbb{P}\otimes dy\text{-a.e.} \quad \forall t \in [0,T].$$

Since the BSPDE (2.20) has a unique solution we also obtain

$$(\tilde{v}^N, \tilde{\zeta}^N) = (\bar{v}^N, \bar{\zeta}^N) \quad \text{in } \mathcal{H}^1 \times \mathcal{L}^2_{\mathcal{F}}(0,T;H^{1,2}(\mathbb{R}^d)).$$

The regularity properties of $\bar{v}^N$ imply that $\tilde{u}^N_t(y)$, $v^N_t(y)$ and $\tilde{v}^N_t(y)$ are all continuous in (t,y) with probability 1. In view of the proof of Theorem 2.4.3, we have $\{\tilde{v}^N_t(y)\}$ converges increasingly to $\theta(y)\theta^{-1}v_t(y^y_t)$ for every $(t,y) \in [0,T] \times \mathbb{R}^d$ with probability 1, as N goes to infinity. Setting

$$(\tilde{v}_t(y), \tilde{\zeta}_t(y)) := \theta(y)((\theta^{-1}v_t)(y^y_t), (\theta^{-1}\zeta_t)(y^y_t)),$$

we obtain $(\tilde{v}, \tilde{\zeta})1_{[0,\tau]} \in \left(\mathcal{H}^1 \cap \mathcal{L}^p_{\mathcal{F}}(0,T;H^{2,p}(\mathbb{R}^d))\right) \times \mathcal{L}^2_{\mathcal{F}}(0,T;H^{1,2}(\mathbb{R}^d))$ for all $\tau \in (0,T)$ and $p \in (2,\infty)$, and

$$\frac{c_0\theta(y)}{T-t} \le \tilde{v}_t(y) \le \frac{c_1\theta(y)}{T-t}, \quad \mathbb{P} \otimes dt \otimes dy\text{-a.e.}$$

Moreover, for every $\tau \in (0,T)$, it holds almost surely

$$\begin{aligned} \tilde{v}_t(y) = \tilde{v}_\tau(y) + \int_t^\tau \Big\{ &\operatorname{tr}\left(\bar{a}_s(y^y_s)D^2\tilde{v}_s(y)\right) + \bar{b}^{\mathcal{T}}_s(y)D\tilde{v}_s(y) + \bar{c}_s(y)\tilde{v}_s(y) \\ &+ \theta(y)F(s, y^y_s, (\theta^{-1}\tilde{v}_s)(y))\Big\}\, ds - \int_t^\tau \tilde{\zeta}_s(y)\, dW_s, \quad dy\text{-a.e.} \end{aligned}$$

Again, by [DQT12, Propostion 6.4], we further have

$$\tilde{v} \in \mathcal{S}^p_{\mathcal{F}}([0,\tau];H^{1,p}(\mathbb{R}^d)) \cap \mathcal{L}^p_{\mathcal{F}}(0,\tau;H^{2,p}(\mathbb{R}^d)), \quad p \in (2,+\infty),$$

and thus, by Sobolev embedding theorem, $\tilde{v} \in \mathcal{S}^p_{\mathcal{F}}([0,\tau];C^\delta(\mathbb{R}^d))$ for every $\delta \in (0,1)$. Therefore, both $\tilde{v}_t(y)$ and $u_t(y) = \theta^{-1}(y - \int_0^t \sigma_s dW_s)\tilde{v}_t(y - \int_0^t \sigma_s dW_s)$ are almost surely continuous in $(t,y) \in [0,\tau] \times \mathbb{R}^d$. Hence,

$$V(t,y,x) := u_t(y)x^2, \quad (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d,$$

coincides with the value function of (2.2) for every $y \in \mathbb{R}^d$. $\square$

3. Smooth Solutions to Portfolio Liquidation Problems under Price-Sensitive Market Impact

We denote by $C(\mathbb{R}^d)$ the space of *bounded* continuous functions. The functions in $u \in C_{poly}([0,T] \times \mathbb{R}^d)$ are continuous and for some $C > 0$ and $n \geq 1$,

$$|u(t,y)| \leq C(1 + |y|^n), \qquad (t,y) \in [0,T] \times \mathbb{R}^d. \tag{3.1}$$

The space $C^{1,2}_{loc}([0,T] \times \mathbb{R}^d)$ denotes the class of the functions $u(t,y)$ that are continuous (possibly unbounded) along with their first derivative in t and their first and second derivative in y. For generic $\alpha \in (0,1)$ and normed vector space E, the functions in $C^{k+\alpha}([0,T];E)$ have α-Hölder continuous derivatives up to the order k. By $W^2_{q,loc}(\mathbb{R}^d)$ we denote the usual Sobolev spaces of all functions that are locally L^q-integrable along with their weak first and second order derivative [AF03, Definition 1.62]. The parabolic version $W^{1,2}_{q,loc}((0,T) \times \mathbb{R}^d)$ is the space of all functions $u(t,y)$ that are locally L^q-integrable along with their weak first derivative in t and their weak first and second derivative in y. Whenever the notation T^- appears in the definition of a function space we mean the intersection over the collection of all the corresponding function spaces when T^- is replaced by $s < T$, e.g.,

$$C_{poly}([0,T^-] \times \mathbb{R}^d) = \bigcap\nolimits_{s<T} C_{poly}([0,s] \times \mathbb{R}^d).$$

The set of adapted $\mathbb{R}^d$-valued processes $(Z_t)_{t\in[0,T]}$ satisfying $E[\int_0^T |Z_t|^q \, dt] < \infty$ is denoted by $L^q_{\mathcal{F}}(0,T;\mathbb{R}^d)$; the subset of processes with continuous paths such that $E[\sup_{t\in[0,T]} |Z_t|^q] < \infty$ is denoted by $L^q_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^d))$. If not otherwise indicated then $\|\cdot\|$ denotes the supremum norm. For arbitrary $\beta > 0$ we occasionally write $\sqrt[\beta]{\cdot}$ instead of $(\,\cdot\,)^{1/\beta}$. All equations are to be understood in the a.s. sense.

This chapter is organized as follows. The stochastic control problem is formulated in Section 3.1. The a priori estimates and the comparison principle that yields uniqueness of a continuous viscosity solution is established in Section 3. Existence and uniqueness of a classical solution to the HJB equation is proven in Section 4. The verification argument is carried out in Section 5. Finally, we show in Section 6 how our uniqueness result extends to the non-Markovian case analyzed in [AJK14].

3.1. Problem formulation and main results

We consider the stochastic optimization problem of an investor that needs to close a (large) position of shares within a given time interval $[0,T]$. Following Horst & Naujokat [HN14] and Kratz & Schöneborn [KS15] the investor may trade in an absolutely continuous manner in a primary exchange and simultaneously place passive block orders into a dark pool. Execution of passive orders is modeled by the jump times of a Poisson process $(N_t)_{t\in[0,T]}$ with constant intensity $\theta \geq 0$.

The Poisson process N and an n-dimensional standard Brownian motion $W = (W_t)_{t\in[0,T]}$ are defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions. In what follows we repeatedly use the independence of N and W.

As the factor process driving trading costs we consider the d-dimensional Itô diffusion

$$Y_s^{t,y} = y + \int_t^s b(Y_r^{t,y})\, dr + \int_t^s \sigma(Y_r^{t,y})\, dW_r, \qquad t \leq s \leq T. \tag{3.2}$$

Assumption 3.1.1. We assume throughout that the coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ are Lipschitz continuous.

The preceding assumption guarantees that the SDE (3.2) admits a unique solution $(Y_s^{t,y})_{s\in[t,T]}$, for every initial state $(t, y) \in [0, T] \times \mathbb{R}^d$. Furthermore, see [PR14, Theorem 3.35], the map $(s, t, y) \mapsto Y_s^{t,y}$ is a.s. continuous and for every $n \geq 2$ there exists $C_n > 0$ such that the following moment estimate holds with the convention $Y_s^{t,y} = y$ for $0 \leq s \leq t$:

$$E[\sup\nolimits_{s\in[0,T]} |Y_s^{t,y}|^n] \leq C_n(1 + |y|^n), \qquad (t, y) \in [0, T] \times \mathbb{R}^d. \tag{3.3}$$

This moment estimate in particular guarantees by Vitali's convergence theorem that functions of the form

$$(t, y) \mapsto E\left[\int_t^T f(s, Y_s^{t,y})\, ds\right], \tag{3.4}$$

with $f \in C_{poly}([0, T] \times \mathbb{R}^d)$, belong again to $C_{poly}([0, T] \times \mathbb{R}^d)$.

3.1.1. The stochastic control problem

For any initial time $t \in [0, T)$ and initial position $x \in \mathbb{R}$, we denote by $\mathcal{A}(t, x)$ the set of all admissible liquidation strategies (ξ, π). Here, $\xi = (\xi_s)_{s\in[t,T]}$ describes the rates at which the agent trades in the primary market, while $\pi = (\pi_s)_{s\in[t,T]}$ describes the passive orders submitted to the dark pool. A pair of strategies (ξ, π) is admissible if ξ is progressively measurable and π is predictable such that the resulting portfolio process

$$X_s^{\xi,\pi} = x - \int_t^s \xi_r\, dr - \int_t^s \pi_r\, dN_r, \qquad t \leq s \leq T,$$

satisfies the liquidation constraint

$$X_T^{\xi,\pi} = 0. \tag{3.5}$$

The costs associated with an admissible liquidation strategy (ξ, π) are modeled by the cost functional

$$J(t, y, x; \xi, \pi) := E\left[\int_t^T \eta(Y_s^{t,y})|\xi_s|^p + \theta\gamma(Y_s^{t,y})|\pi_s|^p + \lambda(Y_s^{t,y})|X_s^{\xi,\pi}|^p \, ds\right].$$

The first term of the nonnegative running costs

$$c(y, x, \xi, \pi) := \eta(y)|\xi|^p + \theta\gamma(y)|\pi|^p + \lambda(y)|x|^p$$

describes the temporary price impact at the primary exchange; the second term describes adverse selection costs associated with dark pool trading (see [HN14, KSS14] for details) while the third term penalizes slow liquidation. The latter may be interpreted as the p-th power of the Value-at-Risk of the open position (see [AK12a, GS11] for details).

Assumption 3.1.2. We assume throughout that $p > 1$[1] and put $\beta := 1/(p-1) > 0$. We further assume that the cost coefficients satisfy the following conditions:

(i) The coefficients $\eta, \gamma, \lambda, 1/\eta : \mathbb{R}^d \to [0, \infty)$ are continuous.

(ii) The coefficients η, $\lambda, 1/\eta$ are of polynomial growth, i.e., for some $n \geq 1$ and $C > 0$,

$$\eta(y) + \lambda(y) + 1/\eta(y) \leq C(1 + |y|^n), \qquad y \in \mathbb{R}^d. \tag{3.6}$$

The value function of the stochastic control problem is defined for each initial state $(t, y, x) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}$ as

$$V(t, y, x) := \inf_{(\xi,\pi)\in\mathcal{A}(t,x)} J(t, y, x; \xi, \pi). \tag{3.7}$$

Example 3.1.3. Our assumptions on the factor process allow us to capture simultaneously several key determinants of trading costs. The assumptions are satisfied for the arithmetic Brownian motion model

$$dY_t^1 = \mu \, dt + \sigma \, dW_t^1$$

as well as for a mean-reverting process of the form

$$dY_t^2 = f(\nu - Y_t^2) \, dt + dW_t^2$$

for a bounded Lipschitz continuous function f. The (logarithmic) price process Y^1 may drive the market risk factor λ while Y^2 may describe stochastic order book heights (stochastic liquidity) and hence drive η.

[1] Unlike [Sch13a] we do not exclude exponents $1 < p < 2$, which correspond to root shaped temporary price impact. Almgren et al. [ATHL05] give empirical evidence for $p = 8/5$.

Remark 3.1.4. We notice that our assumption (A2) below on the diffusion coefficients is not satisfied for an Ornstein-Uhlenbeck process. The semigroup generated by the Ornstein-Uhlenbeck operator in the space of (uniformly) continuous functions is neither strongly continuous nor analytic. Assumptions (A1) and (A2) below are also not satisfied for geometric Brownian motion. Geometric Brownian motion is covered only after an exponential transformation of the cost coefficients provided that the resulting coefficients satisfy the assumptions of Theorem 3.1.8.

3.1.2. Heuristics and the main result

The dynamic programing principle suggests that the value function satisfies the following HJB equation, cf. [SW11, Theorem 2.2]:

$$-\partial_t V(t,y,x) - \mathcal{L}V(t,y,x) - \inf_{\xi,\pi\in\mathbb{R}} H(t,y,x,\xi,\pi,V) = 0, \quad (t,y,x) \in [0,T)\times\mathbb{R}^d\times\mathbb{R}, \tag{3.8}$$

where

$$\mathcal{L} := \frac{1}{2}\operatorname{tr}(\sigma\sigma^* D_y^2) + \langle b, D_y\rangle$$

denotes the infinitesimal generator of the factor process, and the Hamiltonian H is given by

$$H(t,y,x,\xi,\pi,V) := -\xi\partial_x V(t,y,x) + \theta(V(t,y,x-\pi) - V(t,y,x)) + c(y,x,\xi,\pi).$$

The specific structure of our control problem with respect to the state variable x – linear in the control dynamics and of p-th power in the running costs – suggests an ansatz of the form:

$$V(t,y,x) = v(t,y)|x|^p. \tag{3.9}$$

Recalling that $\beta = 1/(p-1)$, the proof of Lemma 3.1.6 below is in fact standard. Before stating the lemma we formulate the different solution concepts for parabolic equations used in this chapter.

Definition 3.1.5. For continuous functions $v : [0,T)\times\mathbb{R}^d \to \mathbb{R}$ we use the following solution concepts to parabolic PDEs

$$-\partial_t v(t,y) - H(t,y,v(t,y),D_y v(t,y),D_y^2 v(t,y)) = 0, \tag{3.10}$$

where $H : [0,T)\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d \to \mathbb{R}$ and $\mathbb{S}^d$ denotes the set of symmetric $d\times d$ matrices.

(i) v is a *classical solution* if $v \in C^{1,2}_{loc}([0,T)\times\mathbb{R}^d)$ such that (3.10) is satisfied for all $(t,y)\in[0,T)\times\mathbb{R}^d$.

(ii) v is a *strong solution* if $v \in W^{1,2}_{1,loc}((0,T)\times\mathbb{R}^d)$ such that (3.10) is satisfied in terms of the weak derivatives of v a.e. in $[0,T)\times\mathbb{R}^d$.

(iii) v is a *viscosity subsolution* if for every $\varphi \in C^{1,2}_{loc}([0,T)\times\mathbb{R}^d)$ such that $\varphi \geq v$ and $\varphi(t,y) = v(t,y)$ at a point $(t,y) \in [0,T)\times\mathbb{R}^d$ it holds

$$-\partial_t\varphi(t,y) - H(t,y,v(t,y),D_y\varphi(t,y),D^2_y\varphi(t,y)) \leq 0.$$

(iv) v is a *viscosity supersolution* if for every $\varphi \in C^{1,2}_{loc}([0,T)\times\mathbb{R}^d)$ such that $\varphi \leq v$ and $\varphi(t,y) = v(t,y)$ at a point $(t,y) \in [0,T)\times\mathbb{R}^d$ it holds

$$-\partial_t\varphi(t,y) - H(t,y,v(t,y),D_y\varphi(t,y),D^2_y\varphi(t,y)) \geq 0.$$

(v) v is a *viscosity solution* if v is both viscosity sub- and supersolution.

Lemma 3.1.6. *A nonnegative function $v : [0,T)\times\mathbb{R}^d \to [0,\infty)$ is a classical/strong/viscosity (sub-/super-)solution to*

$$-\partial_t v(t,y) - \mathcal{L}v(t,y) - F(y,v(t,y)) = 0, \tag{3.11}$$

where

$$F(y,v) := \lambda(y) - \frac{|v|^{\beta+1}}{\beta\eta(y)^\beta} + \frac{\theta\gamma(y)v}{\sqrt[\beta]{\gamma(y)^\beta + |v|^\beta}} - \theta v, \tag{3.12}$$

if and only if $v(t,y)|x|^p$ is a classical/strong/viscosity (sub-/super-)solution to the HJB equation. In this case the infimum in (3.8) *is attained at*

$$\xi^*(t,y,x) = \frac{v(t,y)^\beta}{\eta(y)^\beta}x \quad \text{and} \quad \pi^*(t,y,x) = \frac{v(t,y)^\beta}{\gamma(y)^\beta + v(t,y)^\beta}x \tag{3.13}$$

and

$$H(t,y,x,\xi^*(t,y,x),\pi^*(t,y,x),v(\cdot,\cdot)|\cdot|^p) = F(y,v(t,y))|x|^p. \tag{3.14}$$

To guarantee the uniqueness of a viscosity solution to (3.11) we need to impose a suitable terminal condition. Due to the liquidation constraint (3.5), we expect the value function to tend to infinity for any fixed non-trivial portfolio position as $t \to T$. More precisely, when disregarding any adverse selection and risk costs, as well as any scenarios in which passive orders are executed, and using $\mathbb{P}(\text{no jumps of } N \text{ in } [t,T]) = e^{-\theta(T-t)}$, one obtains for any admissible control $(\xi,\pi) \in \mathcal{A}(t,x)$,

$$J(t,y,x;\xi,\pi) \geq e^{-\theta(T-t)}E\left[\int_t^T \eta(Y_s^{t,y})|\xi_s|^p\,ds \,\middle|\, \text{no jumps of } N \text{ in } [t,T]\right]. \tag{3.15}$$

Applying the reverse Hölder inequality to the inner integral of the RHS of (3.15) yields,

$$J(t,y,x;\xi,\pi) \geq e^{-\theta(T-t)}E\left[\left(\int_t^T \eta(Y_s^{t,y})^{\frac{-1}{p-1}}\,ds\right)^{-(p-1)}\left(\int_t^T |\xi_s|\,ds\right)^p \,\middle|\, \text{no jump}\right].$$

Given that no jumps occur, the liquidation constraint (3.5) yields $\int_t^T \xi_s = x$. Using the independence of $Y^{t,y}$ and N we hence obtain,

$$
\begin{aligned}
J(t,y,x;\xi,\pi) &\geq e^{-\theta(T-t)} E\left[\left(\int_t^T \eta(Y_s^{t,y})^{\frac{-1}{p-1}}\,ds\right)^{-(p-1)} \left|\int_t^T \xi_s\,ds\right|^p \middle|\, \text{no jump}\right] \\
&= E\left[\frac{e^{-\theta(T-t)}}{\sqrt[\beta]{\int_t^T \frac{1}{\eta(Y_s^{t,y})^\beta}\,ds}}\right] |x|^p.
\end{aligned}
$$

In view of (3.6) and (3.3), we therefore expect that

$$
\lim_{t\to T} v(t,y) = +\infty \qquad \text{locally uniformly on } \mathbb{R}^d.
$$

It turns out that this singular terminal condition along with the standing assumptions on the diffusion and cost coefficients already ensures uniqueness of a viscosity solution to the HJB equation. The deterministic closing strategy that liquidates at a constant rate and uses no dark pool incurs the cost

$$
\begin{aligned}
J\Big(t,y,x;\frac{x}{T-t},0\Big) &= \frac{1}{(T-t)^p} E\left[\int_t^T \eta(Y_s^{t,y}) + (T-s)^p \lambda(Y_s^{t,y})\,ds\right] |x|^p \\
&\leq \frac{C}{(T-t)^{p-1}}(1+|y|^n)|x|^p,
\end{aligned}
\tag{3.16}
$$

due to Assumption 3.1.2. As a result, we also expect $v(t,\cdot)$ to satisfy a polynomial growth condition. More precisely, we have the following result; its proof is given in Section 3.2.

Proposition 3.1.7. *Under Assumptions 3.1.1 and 3.1.2 the singular terminal value problem*

$$
\begin{cases}
-\partial_t v(t,y) - \mathcal{L}v(t,y) - F(y,v(t,y)) = 0, & (t,y)\in[0,T)\times\mathbb{R}^d, \\
\lim_{t\to T} v(t,y) = +\infty & \textit{locally uniformly on } \mathbb{R}^d,
\end{cases}
\tag{3.17}
$$

with the nonlinearity F given in (3.12) *admits at most one nonnegative viscosity solution in*

$$
C_{poly}([0,T^-]\times\mathbb{R}^d).
$$

If such a viscosity solution exists, then it satisfies the following a priori estimates for $(t,y)\in[0,T)\times\mathbb{R}^d$:

$$
E\left[\frac{e^{-\theta(T-t)}}{\sqrt[\beta]{\int_t^T \frac{1}{\eta(Y_s^{t,y})^\beta}\,ds}}\right] \leq v(t,y) \leq \frac{1}{(T-t)^p} E\left[\int_t^T \eta(Y_s^{t,y}) + (T-s)^p\lambda(Y_s^{t,y})\,ds\right].
\tag{3.18}
$$

Our main contribution is the proof of the existence of a classical solution to the singular terminal value problem (3.17). This is achieved under the following additional assumptions:

(A1) $\sigma\sigma^*$ is uniformly positive definite.

(A2) η, $1/\eta$, and λ are bounded. In particular, $\eta \geq \kappa_0$ for some constant $\kappa_0 > 0$.

(A3) η is twice continuously differentiable, and b, σ, η', η'' are bounded.

The boundedness away from zero of η, along with the existence and boundedness of η' and η'' will for instance be used to derive the precise asymptotic behavior of the solution near the terminal time. Condition (A1) and the boundedness of b and σ are in particular needed to provide that

$$\begin{cases} D(\mathcal{L}) = \{u \in \bigcap_{q\geq 1} W^2_{q,loc}(\mathbb{R}^d) : u, \mathcal{L}u \in C(\mathbb{R}^d)\} \\ \mathcal{L} : D(\mathcal{L}) \subset C(\mathbb{R}^d) \to C(\mathbb{R}^d) \end{cases} \tag{3.19}$$

is a sectorial realization of the operator $\mathcal{L}$ in $C(\mathbb{R}^d)$ and hence that $\mathcal{L}$ generates an analytic semigroup in $C(\mathbb{R}^d)$, see [Lun95, Corollary 3.1.9]. If $d = 1$, then $D(\mathcal{L}) = C^2(\mathbb{R})$.

We are now ready to state the main result of this chapter. Its proof is carried out in Section 4 below.

Theorem 3.1.8. *Under Assumptions 3.1.1 and 3.1.2 and the conditions (A1), (A2) and (A3), the singular terminal value problem* (3.17) *admits a nonnegative classical solution*

$$v \in C^\alpha([0, T^-]; D(\mathcal{L})) \cap C^{1+\alpha}([0, T^-]; C(\mathbb{R}^d)).$$

Our last result is a verification theorem; its proof is given in Section 3.4. The singularity at the terminal time prevents a straightforward application of the standard verification arguments. Instead, we first use Krylov's generalized Itô formula [Kry80, Theorem 2.10.1, p. 122] to establish optimality away from the terminal time and then use the a priori estimate (3.18) to prove optimality on the whole time interval. Here we require that the strong solution is locally L^q-integrable along with its weak derivatives for some $q > d+2$ to guarantee that the parabolic Sobolev embedding theorem [LSU68, Lemma II.3.3] applies.

Proposition 3.1.9. *Under Assumptions 3.1.1 and 3.1.2 and the conditions (A1) and (A2), for some $q > d + 2$, let*

$$v \in W^{1,2}_{q,loc}((0, T^-) \times \mathbb{R}^d) \cap C_{poly}([0, T^-] \times \mathbb{R}^d)$$

be a nonnegative strong solution to (3.17). *Then, the value function of the control problem* (3.7) *is given by $V(t, y, x) = v(t, y)|x|^p$, and the optimal control (ξ^*, π^*) is* *given in feedback form by*

$$\xi_s^* = \frac{v(s, Y_s^{t,y})^\beta}{\eta(Y_s^{t,y})^\beta} X_s^* \quad \text{and} \quad \pi_s^* = \frac{v(s, Y_s^{t,y})^\beta}{\gamma(Y_s^{t,y})^\beta + v(s, Y_s^{t,y})^\beta} X_{s-}^*. \tag{3.20}$$

In particular, the resulting optimal portfolio process $(X_s^*)_{s\in[t,T]}$ *is given by*

$$X_s^* = x \exp\left(-\int_t^s \frac{v(r, Y_r^{t,y})^\beta}{\eta(Y_r^{t,y})^\beta} \, dr \right) \prod_{t<r\leq s}^{\Delta N_r \neq 0} \left(1 - \frac{v(t, Y_r^{t,y})^\beta}{\gamma(Y_r^{t,y})^\beta + v(t, Y_r^{t,y})^\beta} \right). \tag{3.21}$$

We immediately obtain the following corollary to Theorem 3.1.8 and Proposition 3.1.9.

Corollary 3.1.10. *Assume that the assumptions of Theorem 3.1.8 hold and let v be the unique classical solution to the singular terminal value problem* (3.17)*. Then the optimal liquidation strategy is given in feedback form by (3.20).*

3.2. Comparison principle and a priori estimates

In this section we prove Proposition 3.1.7. The proof is based on the following comparison principle that is itself a consequence of the comparison principle given in the Appendix for viscosity sub- and supersolutions to parabolic equations with *finite* terminal values and monotone nonlinearities.

Lemma 3.2.1. *Let $\underline{v}, \overline{v} \in C_{poly}([0, T^-] \times \mathbb{R}^d)$ be a nonnegative viscosity sub- and a nonnegative viscosity supersolution to* (3.17)*, respectively, such that*

$$\lim_{t\to T} \overline{v}(t, y) = +\infty \quad \text{locally uniformly on } \mathbb{R}^d.$$

Then,

$$\underline{v} \leq \overline{v} \qquad \text{in} \quad [0, T) \times \mathbb{R}^d.$$

In particular, there exists in $C_{poly}([0, T^-] \times \mathbb{R}^d)$ at most one nonnegative viscosity solution to problem (3.17)*.*

Proof. Due to the time-homogeneity of the PDE in (3.17), viscosity (super-/sub-)solutions stay viscosity (super-/sub-)solutions when shifted in time. The idea is therefore to separate the singularities to have finite values to compare.

More precisely, we define, for any $\delta > 0$, the difference function $w : [0, T - \delta) \times \mathbb{R}^d \to \mathbb{R}$ by

$$w(t, y) = \overline{v}(t + \delta, y) - \underline{v}(t, y).$$

A direct computation shows that $v \mapsto F(\cdot, v)$ is decreasing on $[0, \infty)$. In fact, both $\partial_v F$ and $\partial_v^2 F$ are nonpositive on $\mathbb{R}^d \times [0, \infty)$. Hence, by Lemma A.2.2, w is on $[0, T - \delta) \times \mathbb{R}^d$ a viscosity supersolution to

$$-w_t(t, y) - \mathcal{L}w(t, y) - l(t, y)w(t, y) = 0,$$

where

$$l(t,y) := 1_{\overline{v}(t+\delta,y)\neq\underline{v}(t,y)}\frac{F(y,\overline{v}(t+\delta,y)) - F(y,\underline{v}(t,y))}{\overline{v}(t+\delta,y) - \underline{v}(t,y)}.$$

By the first order Taylor approximation of F in v at $\overline{v}(t+\delta,y)$, along with $\partial_v F \leq 0$ and $\partial_v^2 F \leq 0$, we obtain that

$$-l(t,y)w(t,y) \leq -\partial_v F(y,\overline{v}(t+\delta,y))w^+(t,y).$$

In terms of the continuous coefficient

$$\tilde{l}(t,y) := \partial_v F(y,\overline{v}(t+\delta,y)) \leq 0,$$

it follows that w is on $[0,T-\delta)\times\mathbb{R}^d$ also a viscosity supersolution to the equation

$$-\partial_t w(t,y) - \mathcal{L}w(t,y) - \tilde{l}(t,y)w^+(t,y) = 0, \tag{3.22}$$

for which the assumptions of Theorem A.2.1 hold with $\mu = 0$. Hence, for all $0 < t \leq s < T-\delta$ and $y \in \mathbb{R}^d$ Theorem A.2.1 yields

$$w(t,y) \geq E\left[w(s,Y_s^{t,y})\exp\left(\int_t^s \tilde{l}(r,Y_r^{t,y})\,dr\right)\right], \tag{3.23}$$

where the RHS is the Feynman-Kac viscosity solution [Par99, Theorem 3.2] to $-\partial_t w - \mathcal{L}w - \tilde{l}w = 0$ on $[0,s]\times\mathbb{R}^d$, and thus a viscosity subsolution to (3.22) with terminal value $w(s,\cdot)$.

Since $\overline{v} \geq 0$, $\tilde{l} \leq 0$, and $E[\sup_{s\in[t,T-\delta]}\underline{v}(s,Y_s^{t,y})] < \infty$ due to $\underline{v} \in C_{poly}([0,T-\delta]\times\mathbb{R}^d)$, we can apply Fatou's lemma to the expectation in (3.23) as $s \to T-\delta$ to obtain

$$w(t,y) \geq E\left[\liminf_{s\to T-\delta} w(s,Y_s^{t,y})\exp\left(\int_t^s \tilde{l}(r,Y_r^{t,y})\,dr\right)\right]. \tag{3.24}$$

The sample paths of $(Y_s^{t,y})_{s\in[t,T]}$ are bounded a. s., and hence,

$$\lim_{s\to T-\delta} w(s,Y_s^{t,y}) = \lim_{s\to T-\delta}\overline{v}(s+\delta,Y_s^{t,y}) - \underline{v}(s,Y_s^{t,y}) = +\infty \quad \text{a.s.}$$

because $\lim_{s\to T-\delta}\overline{v}(s+\delta,\cdot) = +\infty$ uniformly on compact sets and $\underline{v} \in C_{poly}([0,T-\delta]\times\mathbb{R}^d)$. The limit inferior in (3.24) is therefore a.s. nonnegative. Hence,

$$\overline{v}(t+\delta,y) - \underline{v}(t,y) \geq 0.$$

Finally, by letting $\delta \to 0$ we conclude $\overline{v} - \underline{v} \geq 0$ on $[0,T)\times\mathbb{R}^d$ by continuity of $\overline{v}$. $\square$

The comparison principle establishes the uniqueness statement in Proposition 3.1.7. It also allows us to establish the a priori estimates (3.18).

Proof of Proposition 3.1.7. We show that the lower (upper) estimate in (3.18)—denoted in the following by $\underline{v}$ (respectively, $\overline{v}$)—is a viscosity subsolution (supersolution) to (3.17). The assertion then follows from the comparison principle established in the preceding lemma.

First, note that $\underline{v} \in C_{poly}([0,T^-] \times \mathbb{R}^d)$. In fact, by Jensen's inequality

$$0 \leq \underline{v}(t,y) = \frac{e^{-\theta(T-t)}}{(T-t)^{1/\beta}} E\left[\left(\frac{1}{T-t}\int_t^T \frac{1}{\eta(Y_s^{t,y})^\beta}\,ds\right)^{-1/\beta}\right]$$
$$\leq \frac{e^{-\theta(T-t)}}{(T-t)^{(\beta+1)/\beta}} E\left[\int_t^T \eta(Y_s^{t,y})\,ds\right],$$

and hence the polynomial growth of $\underline{v}$ in y uniformly away from the terminal time follows from (3.3) and the polynomial growth of η. The continuity along with the polynomial growth of $1/\eta$ guarantees continuity of $\underline{v}$, due to Vitali's convergence theorem as pointed out in (3.4).

To establish the subsolution property of $\underline{v}$, let $\varphi \geq \underline{v}$ be a smooth test function on $[0,T) \times \mathbb{R}^d$ such that $\varphi(t,y) = \underline{v}(t,y)$ for some $(t,y) \in [0,T) \times \mathbb{R}^d$. Moreover, we define the stopping time $\tau = \inf\{s \in [t,T] : |Y_s^{t,y} - y| \geq 1\}$. By uniqueness $Y_r^{t,y} = Y_r^{s,Y_s^{t,y}}$ for all $t \leq s \leq r \leq T$. Hence, by the definition of $\underline{v}$ and the Markov property of $Y^{t,y}$,

$$\underline{v}(s\wedge\tau, Y_{s\wedge\tau}^{t,y}) = E\left[\left.\frac{e^{-\theta(T-(s\wedge\tau))}}{\sqrt[\beta]{\int_{s\wedge\tau}^T \eta(Y_r^{t,y})^{-\beta}\,dr}}\,\right|\,\mathcal{F}_{s\wedge\tau}\right].$$

Because $\varphi \geq \underline{v}$, $\varphi(t,y) = \underline{v}(t,y)$, and the tower rule, it holds for $t < s < T$,

$$0 \leq E[\varphi(s\wedge\tau, Y_{s\wedge\tau}^{t,y}) - \underline{v}(s\wedge\tau, Y_{s\wedge\tau}^{t,y})]$$
$$= E[\varphi(s\wedge\tau, Y_{s\wedge\tau}^{t,y}) - \varphi(t,y)] - E\left[\frac{e^{-\theta(T-(s\wedge\tau))}}{\sqrt[\beta]{\int_{s\wedge\tau}^T \eta(Y_r^{t,y})^{-\beta}\,dr}} - \frac{e^{-\theta(T-t)}}{\sqrt[\beta]{\int_t^T \eta(Y_r^{t,y})^{-\beta}\,dr}}\right].$$

Dividing by $s-t$, using Itô's formula and the integral representation of increments of the function $r \mapsto \frac{e^{-\theta(T-r)}}{\sqrt[\beta]{\int_r^T \eta(Y_u^{t,y})^{-\beta}\,du}}$, and noticing that the stochastic integral being stopped at τ is a true martingale, we obtain

$$0 \leq E\left[\frac{1}{s-t}\int_t^{s\wedge\tau} \partial_t\varphi(r,Y_r^{t,y}) + \mathcal{L}\varphi(r,Y_r^{t,y})\,dr\right]$$
$$- E\left[\frac{1}{s-t}\int_t^{s\wedge\tau} \frac{1}{\beta\eta(Y_r^{t,y})^\beta}\frac{e^{-\theta(T-r)}}{\left(\int_r^T \eta(Y_u^{t,y})^{-\beta}\,du\right)^{(\beta+1)/\beta}}\,dr\right]$$
$$- E\left[\frac{1}{s-t}\int_t^{s\wedge\tau} \frac{\theta e^{-\theta(T-t)}}{\sqrt[\beta]{\int_r^T \eta(Y_u^{t,y})^{-\beta}\,du}}\,dr\right].$$

By Jensen's inequality, for $r \in [t, s]$,

$$\sqrt[\beta]{\frac{1}{\int_r^T \eta(Y_u^{t,y})^{-\beta}\,du}} \leq \sqrt[\beta]{\frac{1}{(T-r)^2}\int_r^T \eta(Y_u^{t,y})^{\beta}\,du} \leq \frac{1}{\sqrt[\beta]{T-s}} \sup_{t\leq u\leq T} \eta(Y_u^{t,y}).$$

Since η is of polynomial growth it follows from the equation (3.3) that we can apply the dominated convergence theorem to the third term above when letting $s \to t$. Since $|Y_r^{t,x} - y| \leq 1$ for $r \in [t, \tau]$ and because η^{-1} is bounded on compact domains similar arguments show that we can apply the dominated convergence theorem also to the second term. As $\tau > t$, the fundamental theorem of calculus yields

$$0 \leq \partial_t \varphi(t,y) + \mathcal{L}\varphi(t,y) - \frac{1}{\beta\eta(y)^\beta} E\left[\left(\frac{e^{-\theta(T-t)/(\beta+1)}}{\sqrt[\beta]{\int_t^T \eta(Y_u^{t,y})^{-\beta}\,du}}\right)^{\beta+1}\right] - \theta\underline{v}(t,y).$$

Using Jensen's inequality and $\beta + 1 > 1$ we obtain,

$$0 \leq \partial_t \varphi(t,y) + \mathcal{L}\varphi(t,y) - \frac{\underline{v}(t,y)^{\beta+1}}{\beta\eta(y)^\beta} - \theta\underline{v}(t,y).$$

Hence, from the definition (3.12) of F and $\lambda, \gamma, \underline{v} \geq 0$ it is seen,

$$-\partial_t \varphi(t,y) - \mathcal{L}\varphi(t,y) - F(y, \underline{v}(t,y)) \leq 0.$$

Next, we verify the supersolution property of $\overline{v}$. Because η and λ are of polynomial growth it follows $\overline{v} \in C_{poly}([0,T^-] \times \mathbb{R}^d)$ from (3.4). Again an application of Itô's formula and Leibniz's rule similar as above yields that for every smooth test function $\varphi \leq \overline{v}$ on $[0,T) \times \mathbb{R}^d$ such that $\varphi(t,y) = \overline{v}(t,y)$ for some $(t,y) \in [0,T) \times \mathbb{R}^d$,

$$0 \geq \partial_t \varphi(t,y) + \mathcal{L}\varphi(t,y) + \frac{\eta(y)}{(T-t)^p} + \lambda(y) - \frac{p\overline{v}(t,y)}{T-t}.$$

From the definition (3.12) of F, since $\overline{v} \geq 0$,

$$-F(y, \overline{v}(t,y)) \geq -\lambda(y) + \frac{|\overline{v}(t,y)|^{\beta+1}}{\beta\eta(y)^\beta}.$$

Hence, since $p = (\beta+1)/\beta$ and by setting $u(t,y) = (T-t)^{1/\beta}\overline{v}(t,y)/\eta(y)$,

$$\begin{aligned} -\partial_t \varphi(t,y) &- \mathcal{L}\varphi(t,y) - F(y, \overline{v}(t,y)) \\ &\geq \frac{\eta(y)}{(T-t)^p} - \frac{p\overline{v}(t,y)}{T-t} + \frac{|\overline{v}(t,y)|^{\beta+1}}{\beta\eta(y)^\beta} \\ &= \frac{\eta(y)}{(T-t)^{\frac{\beta+1}{\beta}}}\left(1 - \tfrac{(\beta+1)}{\beta}u(t,y) - \tfrac{1}{\beta}|u(t,y)|^{\beta+1}\right). \end{aligned} \tag{3.25}$$

Using the fact that the map $u \mapsto 1 - \frac{\beta+1}{\beta}u + \frac{1}{\beta}|u|^{\beta+1}$ is nonnegative on $\mathbb{R}$ as it attains its minimum at 1, we conclude

$$-\partial_t \varphi(t,y) - \mathcal{L}\varphi(t,y) - F(t, \overline{v}(t,y)) \geq 0. \qquad \square$$

We close this section with a further application of the comparison principle. Under the conditions (A2) and (A3) it establishes the precise asymptotic behavior of a viscosity solution at the terminal time. This observation will be the starting point of the next section.

Corollary 3.2.2. *Let $v \in C_{poly}([0, T^-] \times \mathbb{R}^d)$ be a nonnegative viscosity solution to problem* (3.17)*. If the assumptions (A2) and (A3) hold, then v satisfies the following asymptotic behavior:*

$$(T-t)^{1/\beta} v(t,y) = \eta(y) + O(T-t) \quad \textit{uniformly in } y \textit{ as } t \to T. \tag{3.26}$$

Proof. The statement is proved by identifying a sub- and a supersolution with the desired asymptotics. Due to (A3), the quantity $\|\mathcal{L}\eta\|$ is well-defined and finite, hence $\delta := \kappa_0 / \|\mathcal{L}\eta\| > 0$. We verify below that

$$\check{v}(t,y) := \frac{\eta(y) - \|\mathcal{L}\eta\|(T-t)}{e^{\theta(T-t)}(T-t)^{1/\beta}} \quad \text{and} \quad \hat{v}(t,y) := \frac{\eta(y) + \frac{1}{2}\|\mathcal{L}\eta\|(T-t)}{(T-t)^{1/\beta}} + (T-t)\|\lambda\|$$

are a nonnegative classical sub- and supersolution to (3.17) on $[T-\delta, T) \times \mathbb{R}^d$, respectively, where nonnegativity follows from $\eta \geq \kappa_0 > 0$ by (A2). Hence, (3.26) follows from the comparison principle.

Specifically, let us fix $(t,y) \in [T-\delta, T) \times \mathbb{R}^d$. To verify the supersolution property of $\hat{v}$, we first obtain by a direct computation,

$$-\partial_t \hat{v}(t,y) - \mathcal{L}\hat{v}(t,y) = -\frac{\eta(y) + \frac{1-\beta}{2}\|\mathcal{L}\eta\|(T-t) + \beta\mathcal{L}\eta(T-t)}{\beta(T-t)^{(\beta+1)/\beta}} + \|\lambda\|. \tag{3.27}$$

Recalling the definition (3.12) of F, we have since $\hat{v} \geq 0$,

$$-F(y, \hat{v}(t,y)) \geq -\lambda(y) + \frac{|\hat{v}(t,y)|^{\beta+1}}{\beta\eta(y)^\beta}.$$

Next, we apply Bernoulli's inequality in the form $(u+v+w)^{\beta+1} \geq u^{\beta+1}(1+v/u)^{\beta+1} \geq u^{\beta+1} + (\beta+1)u^\beta v$ for $u, v, w \geq 0$ to the term $|\hat{v}(t,y)|^{\beta+1}$ and obtain

$$-F(y, \hat{v}(t,y)) \geq -\lambda(y) + \frac{\eta(y)^{\beta+1} + (\beta+1)\eta(y)^\beta \frac{1}{2}\|\mathcal{L}\eta\|(T-t)}{\beta\eta(y)^\beta (T-t)^{(\beta+1)/\beta}}. \tag{3.28}$$

Hence, adding (3.27) and (3.28) yields,

$$-\partial_t \hat{v}(t,y) - \mathcal{L}\hat{v}(t,y) - F(y, \hat{v}(t,y)) \geq \|\lambda\| - \lambda(y) + \frac{\|\mathcal{L}\eta\| - \mathcal{L}\eta(y)}{(T-t)^{1/\beta}} \geq 0.$$

Next, we verify the subsolution property of $\check{v}$. By a direct computation

$$-\partial_t \check{v}(t,y) - \mathcal{L}\check{v}(t,y) = -\frac{\eta(y) + (\beta-1)\|\mathcal{L}\eta\|(T-t) + \beta\mathcal{L}\eta(y)(T-t)}{\beta e^{\theta(T-t)}(T-t)^{(\beta+1)/\beta}} - \theta\check{v}_t(t,y). \tag{3.29}$$

On the other hand, since $\lambda, \gamma \geq 0$, and $\check{v} \geq 0$ on $[T-\delta) \times \mathbb{R}^d$,

$$-F(y, \check{v}(t,y)) \leq \frac{|\check{v}(t,y)|^{\beta+1}}{\beta \eta(y)^\beta} + \theta \check{v}(t,y).$$

This time, recalling that δ is chosen such that $\eta(y) \geq \|\mathcal{L}\eta\|(T-t)$, we estimate the term $|\check{v}(t,y)|^{\beta+1}$ by the fact $(u-v)^{\beta+1} = u^{\beta+1}(1-v/u)^{\beta+1} \leq u^{\beta+1} - u^\beta v$ for $u \geq v \geq 0$ and obtain

$$-F(y, \check{v}(t,y)) \leq \frac{\eta(y) - \|\mathcal{L}\eta\|(T-t)}{\beta e^{\theta(\beta+1)(T-t)}(T-t)^{(\beta+1)/\beta}} + \theta \check{v}(t,y). \tag{3.30}$$

Finally, adding (3.29) and (3.30), and using $\beta > 0$ yields,

$$-\partial_t \check{v}(t,y) - \mathcal{L}\check{v}(t,y) - F(t, \check{v}(t,y)) \leq 0. \qquad \square$$

3.3. Existence of a classical solution

In this section we prove Theorem 3.1.8 and hence assume throughout that (A1), (A2) and (A3) hold. Our existence proof is based on the explicit asymptotic behavior established in Corollary 3.2.2. It tells us the solution must be of the form

$$v(T-t,y) = \frac{\eta(y) + \tilde{u}(t,y)}{t^{1/\beta}}, \quad \tilde{u}(t,y) = O(t) \text{ uniformly in } y \text{ as } t \to 0, \tag{3.31}$$

where we reversed the time variable as we will do for the rest of this subsection. For reasons that will become clear later, it will be more convenient to choose the following equivalent ansatz:

$$v(T-t,y) = \frac{\eta(y)}{t^{1/\beta}} + \frac{u(t,y)}{t^{1+1/\beta}}, \quad u(t,y) = O(t^2) \text{ uniformly in } y \text{ as } t \to 0. \tag{3.32}$$

Plugging the asymptotic ansatz into (3.17) results in a semilinear parabolic equation for u with finite initial condition, but with a singularity in the nonlinearity. This motivates the following lemma.

Lemma 3.3.1. *If for some $\delta > 0$ a map $u \in C^\alpha([0,\delta]; D(\mathcal{L})) \cap C^{1+\alpha}([0,\delta]; C(\mathbb{R}^d))$ satisfies*

$$|u(t,y)| \leq t\eta(y), \quad t \in [0,\delta],\, y \in \mathbb{R}^d, \tag{3.33}$$

and solves the equation

$$\begin{aligned}\partial_t u(t,y) = \mathcal{L}u(t,y) + t\mathcal{L}\eta(y) + t^p\lambda(y) - \frac{\eta(y)}{\beta}\sum_{k=2}^{\infty}\binom{\beta+1}{k}\left(\frac{u(t,y)}{t\eta(y)}\right)^k \\ + \frac{\theta t^p \gamma(y)(t\eta(y) + u(t,y))}{\sqrt[\beta]{(t^p\gamma(y))^\beta + |t\eta(y)+u(t,y)|^\beta}} - \theta(t\eta(y) + u(t,y)), \quad t > 0\,, y \in \mathbb{R}^d,\end{aligned} \tag{3.34}$$

then a local solution $v \in C^{\alpha}([T-\delta, T^{-}]; D(\mathcal{L})) \cap C^{1+\alpha}([T-\delta, T^{-}]; C(\mathbb{R}^d))$ *to problem* (3.17) *is given by*

$$v(t,y) = \frac{\eta(y)}{(T-t)^{1/\beta}} + \frac{u(T-t,y)}{(T-t)^{1+1/\beta}}.$$

Proof. The statement is verified by plugging the ansatz into (3.17), multiplying by $t^p = t^{(\beta+1)/\beta}$, and by using the binomial series for the term

$$t^p \frac{|v(T-t,y)|^{\beta+1}}{\beta \eta(y)^{\beta}} = \frac{\eta(y)}{\beta} \left| 1 + \frac{u(t,y)}{t\eta(y)} \right|^{\beta+1} = \frac{\eta(y)}{\beta} \sum_{k=0}^{\infty} \binom{\beta+1}{k} \left(\frac{u(t,y)}{t\eta(y)} \right)^k$$

to see that the first two terms of the series cancel out. The growth condition (3.33) guarantees that the binomial series does indeed converge. $\square$

Remark 3.3.2. The reason for choosing the ansatz (3.32) is that the series in (3.34) starts at $k=2$, and not at $k=1$. This will be crucial for the fixed point argument below. The more straightforward ansatz (3.31) results in the equation

$$\partial_t \tilde{u} = \mathcal{L}\tilde{u} + \mathcal{L}\eta + t^{\frac{1}{\beta}} \lambda - \frac{\eta}{\beta t} \left(\frac{\tilde{u}}{\eta} + \sum_{k=2}^{\infty} \binom{\beta+1}{k} \left(\frac{\tilde{u}}{\eta} \right)^k \right) + \frac{\theta t^{\frac{1}{\beta}} \gamma(\eta + \tilde{u})}{\sqrt[\beta]{t\gamma^{\beta} + |\eta + \tilde{u}|^{\beta}}} - \theta(\eta + \tilde{u}),$$

for which we have no analogues to Lemma 3.3.4 and Lemma 3.3.5 below, due to the term $t^{-1}\tilde{u}$.

We will solve equation (3.34) using the semigroup approach for parabolic equations in Banach spaces; we refer to the monograph by Lunardi [Lun95] as the standard reference. To this end, we interpret (3.34) as an evolution equation

$$u'(t) = \mathcal{L}u(t) + f(t, u(t)), \quad t > 0; \quad u(0) = 0, \tag{3.35}$$

in the Banach algebra $U := C(\mathbb{R}^d)$ of bounded continuous functions endowed with the supremum norm $\|\cdot\|$, where the nonlinearity f is given by

$$f(t,u) = t\mathcal{L}\eta + t^p \lambda - \frac{\eta}{\beta} \sum_{k=2}^{\infty} \binom{\beta+1}{k} \left(\frac{u}{t\eta} \right)^k + \frac{\theta t^p \gamma(t\eta + u)}{\sqrt[\beta]{(t^p\gamma)^{\beta} + |t\eta + u|^{\beta}}} - \theta(t\eta + u).$$

The general theory suggests to look first for a local *mild solution* of (3.35). That is, to show there is a fixed point u of the integral operator Γ defined in $C([0,\delta]; U)$ by

$$\Gamma(u)(t) = \int_0^t e^{(t-s)\mathcal{L}} f(s, u(s))\, ds, \quad 0 \le t \le \delta, \tag{3.36}$$

if $\delta > 0$ is small enough where $\{e^{t\mathcal{L}} : t \ge 0\}$ is the analytic semigroup generated by $\mathcal{L}$ in U. Regularity of the mild solution u will then follow from analyticity of the semigroup and Hölder continuity of $t \mapsto f(t, u(t))$.

The singular behavior of f near $t = 0$ prevents us from directly applying general theory. In fact, the operator Γ is not defined on the whole space $C([0,\delta]; U)$, and its domain is not closed with respect to the supremum norm. We overcome these difficulties by carrying out the usual contraction argument with respect to an appropriate weighted norm on $C([0,\delta];U)$.

In order to guarantee that the function $t \mapsto f(t,u(t))$ behaves well at $t=0$ it seems reasonable to restrict the set of potential mild solutions to those functions $u \in C([0,\delta];U)$ such that $u(t)=o(t)$ as $t \to 0$. Yet, there is no nice norm making this set of functions a Banach space. Recalling (3.26) however, we actually expect the slightly stronger condition $u(t)=O(t^2)$ as $t\to 0$ to be satisfied. This suggests to view Γ as an operator acting in the space

$$E = \big\{ u \in C([0,\delta];U) : u(t) = O(t^2) \text{ as } t \to 0 \big\},$$

endowed with the weighted norm

$$\|u\|_E = \sup_{0<t\le\delta} \left\| t^{-2}u(t) \right\|.$$

Lemma 3.3.3. *The vector space E endowed with the norm $\|\cdot\|_E$ is a Banach space.*

The next lemma shows in particular that the integral operator Γ given in (3.36) is well-defined on the closed ball

$$\overline{B}_E(\kappa_0/\delta) := \big\{ u \in E : \|u\|_E \le \kappa_0/\delta \big\}.$$

Lemma 3.3.4. *Let $R>0$ and $\delta \in (0,\kappa_0/R]$.*

(i) For every $u \in \overline{B}_E(R)$, the function $f(\,\cdot\,,u(\,\cdot\,))$ belongs to $C([0,\delta];U)$. In particular, the operator Γ defined in (3.36) *is well defined on $\overline{B}_E(R)$.*

(ii) If $u \in \overline{B}_E(R) \cap C^\alpha([0,\delta];U)$ for some $\alpha \in (0,1)$, then $f(\,\cdot\,,u(\,\cdot\,))$ is α-Hölder continuous, i.e., belongs to $C^\alpha([0,\delta];U)$.

Proof. For $u \in \overline{B}_E(R)$ we consider the functions $g:[0,\delta]\to U$ and $h:[0,\delta]\times U \to U$ given by

$$g(t) = \sum_{k=2}^{\infty} \binom{\beta+1}{k} \left(\frac{u(t)}{t\eta}\right)^k \quad \text{and} \quad h(t,w) = \frac{t^p\gamma w}{\sqrt[\beta]{(t^p\gamma)^\beta + |w|^\beta}} - w,$$

so that we may decompose $f(t,u(t))$ in the following way:

$$f(t,u(t)) = t\mathcal{L}\eta + t^p\lambda - (p-1)\eta g(t) + \theta h(t, t\eta + u(t)). \tag{3.37}$$

The assumption $\delta \le \kappa_0/R$ guarantees that the series defining $g(t)$ converges in U since then

$$\left\| \frac{u(t)}{t\eta} \right\| \le \frac{t^2R}{t\kappa_0} \le \frac{\delta R}{\kappa_0} \le 1, \quad t \in [0,\delta].$$

In view of (3.37) it will be sufficient to show that g and $h(\,\cdot\,, \cdot\,\eta + u(\cdot))$ are continuous, or even α-Hölder continuous if $u \in C^\alpha([0,\delta];U)$. For the latter note that h is continuously differentiable on $(0,\delta] \times U$. In fact,

$$\|\partial_t h(t,w)\|_{L(U)} = \left\| \frac{pt^{p-1}\gamma w|w|^\beta}{\sqrt[\beta]{(t^p\gamma)^\beta + |w|^\beta}^{\beta+1}} \right\| \le \left\| \frac{pt^{p-1}\gamma w|w|^\beta}{(t^p\gamma)^{\beta+1} + pt^p\gamma|w|^\beta} \right\| \le \frac{\|w\|}{t},$$

where we used Bernoulli's inequality and $\beta + 1 = p\beta$, and

$$\|\partial_w h(t,w)\|_{L(U)} = \left\| \frac{(t^p\gamma)^{\beta+1}}{\sqrt[\beta]{(t^p\gamma)^\beta + |w|^\beta}^{\beta+1}} - 1 \right\| \le 1.$$

Hence, for all $0 \le t \le s \le \delta$,

$$\begin{aligned}
&\|h(t, t\eta + u(t)) - h(s, s\eta + u(s))\| \\
&\le \frac{\|t\eta + u(t)\|}{t}|t-s| + \|\eta\||t-s| + \|u(t) - u(s)\| \\
&\le (2\|\eta\| + \kappa_0)|t-s| + \|u(t) - u(s)\|.
\end{aligned}$$

In order to establish the continuity of g, notice that for every $k \ge 2$ and $0 \le t \le s \le \delta$ it holds that

$$\begin{aligned}
\left\| \left(\frac{u(t)}{t\eta}\right)^k - \left(\frac{u(s)}{s\eta}\right)^k \right\| &\le \left\| \left(\frac{u(t)}{t\eta}\right)^k - \left(\frac{u(t)}{s\eta}\right)^k \right\| + \left\| \left(\frac{u(t)}{s\eta}\right)^k - \left(\frac{u(s)}{s\eta}\right)^k \right\| \\
&\le \frac{\|u(t)\|^k}{\kappa_0^k} \left| \frac{1}{t^k} - \frac{1}{s^k} \right| + \frac{1}{s^k\kappa_0^k} \|u(t) - u(s)\| \sum_{l=0}^{k-1} \|u(t)\|^l \|u(s)\|^{k-1-l} \\
&\le \frac{t^{2k}R^k}{t^k s^k \kappa_0^k} \left| t^k - s^k \right| + \frac{R^{k-1}}{s^k\kappa_0^k} \|u(t) - u(s)\| \sum_{l=0}^{k-1} t^{2l} s^{2(k-1-l)} \\
&\le \frac{k\delta^{k-1}R^k}{\kappa_0^k} |t-s| + \frac{k\delta^{k-2}R^{k-1}}{\kappa_0^k} \|u(t) - u(s)\| \\
&\le \frac{kR}{\kappa_0}|t-s| + \frac{kR}{\kappa_0^2} \|u(t) - u(s)\|.
\end{aligned} \tag{3.38}$$

Using the identity $k\binom{\beta+1}{k} = (\beta+1)\binom{\beta}{k-1}$ it follows that

$$\|g(t) - g(s)\| \le \frac{(2^\beta - 1)(\beta+1)R}{\kappa_0}|t-s| + \frac{(2^\beta - 1)(\beta+1)R}{\kappa_0^2}\|u(t) - u(s)\|.$$

Hence, g is uniformly continuous and even α-Hölder continuous if $u \in C^\alpha([0,\delta];U)$. □

The usual assumption on the nonlinearity to carry out the fixed point argument would be that $f(t,u)$ is locally Lipschitz continuous in u uniformly in t. The next lemma proves an appropriate analogue to this assumption for our singular nonlinearity f.

Lemma 3.3.5. *For every $R > 0$ there exists a constant $L > 0$ independent of $\delta \in (0, \kappa_0/R]$ such that*

$$\|f(t, u(t)) - f(t, v(t))\| \leq L \|u(t) - v(t)\|, \quad u, v \in \overline{B}_E(R),\ t \in [0, \delta].$$

Proof. Let $u, v \in \overline{B}_E(R)$ and $t \in [0, T]$. The proof of Lemma 3.3.4 shows that the function h in (3.37) is nonexpanding in the second argument, and estimates similar to those in (3.38) yield

$$\left\| \left(\frac{u(t)}{t\eta}\right)^k - \left(\frac{v(t)}{t\eta}\right)^k \right\| \leq \frac{kR}{\kappa_0^2} \|u(t) - v(t)\|$$

for every $k \geq 2$. Hence, using once more that $k\binom{\beta+1}{k} = (\beta+1)\binom{\beta}{k-1}$ and $(\beta+1)/\beta = p$ we conclude that

$$\|f(t, u(t)) - f(t, v(t))\| \leq (p(2^\beta - 1)\kappa_0^{-2} R\|\eta\| + \theta) \|u(t) - v(t)\|. \qquad \square$$

We are now ready to carry out the fixed point argument and to prove the desired regularity of the fixed point. In view of Lemma 3.3.1 this then gives us a local solution to the problem (3.17).

Proposition 3.3.6. *Under assumptions (A1) and (A2), there exists a short-time solution*

$$u \in C^\alpha([0, \delta]; D(\mathcal{L})) \cap C^{1+\alpha}([0, \delta]; C(\mathbb{R}^d))$$

to the equation (3.34) *that satisfies the growth condition* (3.33).

Proof. We prove below that there exists $R > 0$ and $\delta \in (0, \kappa_0/R]$ such that the operator Γ defined by (3.36) has a fixed point $\overline{u}$ in $\overline{B}_E(R)$.

In order to see that this (local) mild solution to (3.35) belongs to $C^\alpha([0,\delta];D(\mathcal{L}))\cap C^{1+\alpha}([0, \delta]; U)$ notice first that $f(\,\cdot\,, \overline{u}(\,\cdot\,)) \in C([0, \delta]; U)$, due to Lemma 3.3.4(i). Thus, it follows from [Lun95, Proposition 4.2.1] that Γ maps into $C^\alpha([0, \delta]; U)$, for every $\alpha \in (0, 1)$. Since $\overline{u}$ is a fixed point of Γ it follows from Lemma 3.3.4(ii) that $f(\,\cdot\,, \overline{u}(\,\cdot\,)) \in C^\alpha([0, \delta]; U)$.

Moreover, $\overline{u} \in E$ implies that $\overline{u}(0) = \mathcal{L}\overline{u}(0) + f(0, \overline{u}(0)) \equiv 0$ belongs to the domain of $\mathcal{L}$. Along with the Hölder continuity of $f(\,\cdot\,, \overline{u}(\,\cdot\,))$ it now follows from [Lun95, Theorem 7.1.10(iv)] that $\overline{u} \in C^\alpha([0, \delta]; D(\mathcal{L})) \cap C^{1+\alpha}([0, \delta]; U)$ as desired.

It remains to prove the existence of a fixed point of the operator Γ. In terms of $M = \sup_{0 \leq t \leq 1} \|e^{t\mathcal{L}}\|_{L(U)}$ we claim that one can choose

$$R = 2M\,(\|\mathcal{L}\eta\| + \|\lambda\| + \theta\|\eta\|) \quad \text{and} \quad \delta = \min\{\kappa_0/R, (2ML)^{-1}, 1\},$$

where $L > 0$ is the Lipschitz constant given by Lemma 3.3.5. Since $\delta \leq \kappa_0/R$ the operator Γ is well-defined on $\overline{B}_E(R)$, due to Lemma 3.3.4. To show that Γ is a contraction with respect to $\|\cdot\|_E$, let $u, v \in \overline{B}_E(R)$. By the choice of M it holds for every $t \in [0, \delta]$ that

$$\begin{aligned}
\|\Gamma(u)(t) - \Gamma(v)(t)\| &\leq tM \sup_{s\in[0,t]} \|f(s, u(s)) - f(s, v(s))\| \\
&\leq \delta ML \sup_{s\in[0,t]} \|u(s) - v(s)\| \\
&\leq \delta MLt^2 \|u - v\|_E.
\end{aligned}$$

Hence,

$$\|\Gamma(u) - \Gamma(v)\|_E \leq \frac{1}{2}\|u - v\|_E.$$

To show that Γ maps $\overline{B}_E(R)$ into itself, note that since $\delta \leq 1$ and $p > 1$ one has that $s^p \leq s$ for all $s \in [0, \delta]$, and so it holds for every $t \in [0, \delta]$ that

$$\begin{aligned}
\|\Gamma(u)(t)\| &\leq \|\Gamma(u)(t) - \Gamma(0)(t)\| + \|\Gamma(0)(t)\| \\
&\leq t^2\frac{R}{2} + tM \sup_{s\in[0,t]} \left\| s\mathcal{L}\eta + s^p\lambda + \frac{\theta s^p\gamma s\eta}{\sqrt[\beta]{(s^p\gamma)^\beta + (s\eta)^\beta}} - \theta s\eta \right\| \\
&\leq t^2\frac{R}{2} + tM \sup_{s\in[0,t]} \{s\|\mathcal{L}\eta\| + s^p\|\lambda\| + \theta s\|\eta\|\} \\
&\leq t^2 R.
\end{aligned}$$

The operator Γ does therefore map $\overline{B}_E(R)$ contractive into itself. Hence, it has a unique fixed point $\overline{u}$ in $\overline{B}_E(R)$. $\square$

We are now ready to prove Theorem 3.1.8.

Proof of Theorem 3.1.8. In view of Lemma 3.3.1 and Proposition 3.3.6 there exists a (unique) local classical, and hence mild solution

$$u \in C^\alpha([T - \delta, T^-]; D(\mathcal{L})) \cap C^{1+\alpha}([T - \delta, T^-]; C(\mathbb{R}^d))$$

to (3.17). In order to see that the local solution extends to a global solution

$$v \in C^\alpha([0, T^-]; D(\mathcal{L})) \cap C^{1+\alpha}([0, T^-]; C(\mathbb{R}^d))$$

notice first that the functional $v \mapsto F(\cdot, v(\cdot))$ mapping $C(\mathbb{R}^d)$ into itself is continuously differentiable and thus locally Lipschitz continuous. By [Lun95, Corollary 3.1.9], the operator $\mathcal{L}$ generates an analytic semigroup in $C(\mathbb{R}^d)$. Hence, by [Lun95, Theorem 7.1.2] there exists a mild solution $v \in L^\infty(\tau, T - \delta; C(\mathbb{R}^d))$ to $-\partial_t v - \mathcal{L}v - F(y, v) = 0$ for some $0 \leq \tau < T - \delta$ when imposed at $t = T - \delta$ with a terminal value in $C(\mathbb{R}^d)$. Due to the a priori estimates established in Corollary 3.2.2 and [Lun95, Proposition 7.1.8] we may choose $\tau = 0$. This gives us a global mild solution

$$v \in L^\infty(0, T^-; C(\mathbb{R}^d))$$

by pasting v and u at $T-\delta$. In order to verify the desired regularity we recall that F is independent of the time variable. Hence, the regularity follows from [Lun95, Proposition 7.1.10(iv)] if $v(T-\delta,\cdot) \in D(\mathcal{L})$ and if $\mathcal{L}v(T-\delta,\cdot)+F(\cdot,v(T-\delta,\cdot))$ belongs to the real interpolation space $D_{\mathcal{L}}(\alpha,\infty)$. The former condition is a part of the assumption, the later is a consequence of [Lun95, Proposition 2.2.12(i)]. □

3.4. Verification argument

This section is devoted to the verification argument. Throughout, for some $q > d+2$, let

$$v \in W^{1,2}_{q,loc}((0,T^-) \times \mathbb{R}^d) \cap C_{poly}([0,T^-] \times \mathbb{R}^d)$$

denote a nonnegative strong solution to (3.17). We recall that by the parabolic Sobolev embedding theorems [LSU68, Lemma II.3.3], for every $R>0$ the parabolic Sobolev space $W^{1,2}_q((0,T) \times B_d(R))$ is continuous embedded into $C^{l/2,1+l}([0,T] \times \overline{B}_d(R))$ with $l = 1-(d+2)/q$. Here, $B_d(R)$ denotes the d-dimensional ball of radius R centered at the origin and $C^{l/2,1+l}([0,T] \times \overline{B}_d(R))$ denotes the usual parabolic Hölder space of functions $u(t,y)$ on $[0,T] \times \overline{B}_d(R)$ that are $l/2$-Hölder continuous in t and l-Hölder continuous in y along with their first derivative in y. Hence, we assume from now on that Dv is continuous.

The verification argument is established as follows. We first prove that the candidate optimal strategy (ξ^*,π^*) is admissible, and that the resulting portfolio process is monotone. This uses the lower estimate in (3.18). Admissibility does not a priori guarantee that the strategy (ξ^*,π^*) generates finite costs, though. This requires an extra argument.

Subsequently, we show that we may w.l.o.g. restrict ourselves to admissible controls that result in a monotone portfolio process. Similar to [GHQ15, Kra14], we then prove the optimality of ξ^* and π^* in every interval $[t,s]$ with $s<T$. The upper estimate in (3.18) will be used to show that candidate strategy is optimal on the whole time interval.

Lemma 3.4.1. *The pair of feedback controls* (ξ^*,π^*) *given by* (3.20) *is admissible. The portfolio process* $(X^*_s)_{s\in[t,T]}$ *with respect to* (ξ^*,π^*) *is monotone.*

Proof. One readily verifies that the portfolio process $(X^*_s)_{s\in[t,T]}$ with respect to the controls ξ^* and π^* is given by (3.21) and thus is monotone. To show that $X^*_T=0$ we define the random variable

$$\nu(\omega) := \sup_{t\le r\le T}\left\{e^{\beta\theta(T-r)}\eta(Y^{t,y}_r)^\beta E\left[\sup_{r\le u\le T}\eta(Y^{t,y}_u)^{-\beta}\middle|\,\mathcal{F}_r\right]\right\}$$

that is a.s. finite due to Assumption 3.1.2 and the moment estimates (3.3). For $0 \le t \le s < T$, using the lower estimate in (3.18) and Jensen's inequality we obtain,

$$
\begin{aligned}
|X_s^*| &\le |x| \exp\left(- \int_t^s \frac{v(r, Y_r^{t,y})^\beta}{\eta(Y_r^{t,y})^\beta} \, dr \right) \\
&\le |x| \exp\left(- \int_t^s \frac{1}{\eta(Y_r^{t,y})^\beta} E\left[\frac{e^{-\theta(T-r)}}{\sqrt[\beta]{\int_r^T \eta(Y_u^{t,y})^{-\beta} \, du}} \middle| \mathcal{F}_r \right]^\beta dr \right) \\
&\le |x| \exp\left(- \int_t^s \frac{e^{-\beta\theta(T-r)} \eta(Y_r^{t,y})^{-\beta}}{E\left[\int_r^T \eta(Y_u^{t,y})^{-\beta} \, du \middle| \mathcal{F}_r \right]} \, dr \right) \\
&\le |x| \exp\left(-\frac{1}{\nu} \int_t^s \frac{1}{T-r} \, dr \right) = |x| \left(\frac{T-s}{T-t} \right)^{1/\nu} \overset{s \to T}{\longrightarrow} 0.
\end{aligned}
$$

This yields $X_{T-}^* = 0$. Hence, $\pi_T^* = 0$ and so $X_T^* = 0$. □

Lemma 3.4.2. *For every $(\xi, \pi) \in \mathcal{A}(t,x)$ there exists $(\bar{\xi}, \bar{\pi}) \in \mathcal{A}(t,x)$ with lesser or equal costs such that $(X_s^{\bar{\xi},\bar{\pi}})_{s \in [t,T]}$ is monotone.*

Proof. For $(\xi, \pi) \in \mathcal{A}(t,x)$ with $x \ge 0$ consider the strategy $(\bar{\xi}, \bar{\pi})$ given by

$$
\bar{\xi}_s = \xi_s 1_{\{\xi_s \ge 0\}} 1_{\{X_s^{\bar{\xi},\bar{\pi}} > 0\}} \qquad \text{and} \qquad \bar{\pi}_s = (\pi_s \wedge X_{s-}^{\bar{\xi},\bar{\pi}}) 1_{\{\pi_s \ge 0\}} 1_{\{X_{s-}^{\bar{\xi},\bar{\pi}} > 0\}}.
$$

By construction it holds that $0 \le \bar{\xi}_s \le |\xi_s|$, $0 \le \bar{\pi}_s \le |\pi_s|$ and $0 \le X_s^{\bar{\xi},\bar{\pi}} \le |X_s^{\xi,\pi}|$ for all $s \in [t,T]$. As a result, $(X_s^{\bar{\xi},\bar{\pi}})_{s \in [t,T]}$ is monotone decreasing and $X_T^{\bar{\xi},\bar{\pi}} = 0$ by admissibility of (ξ, π). Hence, $(\bar{\xi}, \bar{\pi}) \in \mathcal{A}(t,x)$ with less or equal costs than (ξ, π). The case $x \le 0$ is similar. □

We denote by $\bar{\mathcal{A}}(t,x)$ the set of all admissible controls under which the portfolio process is monotone. For any $(\xi, \pi) \in \bar{\mathcal{A}}(t,x)$ with finite costs the expected residual costs vanish as $s \to T$ as shown by the following lemma.

Lemma 3.4.3. *Under assumption (A2), for every $(\xi, \pi) \in \bar{\mathcal{A}}(t,x)$ with finite costs it holds that*

$$
E\left[v(s, Y_s^{t,y}) |X_s^{\xi,\pi}|^p \right] \longrightarrow 0, \quad s \to T. \tag{3.39}
$$

Proof. The monotonicity of $(X_s^{\xi,\pi})_{s \in [t,T]}$ together with $X_T^{\xi,\pi} = 0$ implies

$$
|x| \ge |X_{s-}^{\xi,\pi}| \ge |X_s^{\xi,\pi}| \ge \left| \int_s^T \pi_r \, dN_r \right| \quad \text{and} \quad |X_{s-}^{\xi,\pi}| \ge |\pi_s| \tag{3.40}
$$

for all $t \le s \le T$, and moreover by Jensen's inequality

$$
|X_s^{\xi,\pi}|^p \le 2^{p-1} \left(\left| \int_s^T \xi_r \, dr \right|^p + \left| \int_s^T \pi_r \, dN_r \right|^p \right). \tag{3.41}
$$

By Itô's formula,

$$\left|\int_s^T \pi_r \, dN_r\right|^p = \int_s^T \left\{ \left|\int_r^T \pi_u \, dN_u + \pi_r\right|^p - \left|\int_r^T \pi_u \, dN_u\right|^p \right\} dN_r.$$

Using once more Jensen's inequality, and then (3.40), we obtain

$$\begin{aligned} \left|\int_s^T \pi_r \, dN_r\right|^p &\le \int_s^T \left\{ (2^{p-1}-1) \left|\int_r^T \pi_u \, dN_u\right|^p + 2^{p-1} |\pi_r|^p \right\} dN_r \\ &\le \int_s^T (2^p - 1) |X_{r-}^{\xi,\pi}|^p \, dN_r. \end{aligned} \tag{3.42}$$

As the integrand in (3.42) is bounded by $(2^p-1)|x|$ its jump integral has a true martingale part. Hence, by decomposing the jump integral in (3.42) into its martingale part with respect to the compensated Poisson process $N_t - \theta t$ and its deterministic part we obtain from (3.41),

$$|X_s^{\xi,\pi}|^p \le 2^{p-1} E\left[\left|\int_s^T \xi_r \, dr\right|^p + \int_s^T (2^p-1)|X_r^{\xi,\pi}|^p \, \theta dr \,\middle|\, \mathcal{F}_s \right],$$

which implies by Gronwall's inequality the existence of a constant $C > 0$ such that

$$|X_s^{\xi,\pi}|^p \le CE\left[\left|\int_s^T \xi_r \, dr\right|^p \,\middle|\, \mathcal{F}_s \right].$$

Next, we apply again Jensen's inequality to obtain,

$$|X_s^{\xi,\pi}|^p \le C(T-t)^{p-1} E\left[\int_s^T |\xi_r|^p \, dr \,\middle|\, \mathcal{F}_s \right].$$

Therefore, by the upper estimate in (3.18) and the boundedness of η and λ due to (A2),

$$\begin{aligned} E\left[v(s, Y_s^{t,y})|X_s^{\xi,\pi}|^p\right] &\le CE\left[\frac{E\left[\int_s^T \eta(Y_r^{t,y}) + (T-r)^p \lambda(Y_r^{t,y}) \, dr \,\middle|\, \mathcal{F}_s\right]}{(T-s)^p} |X_s^{\xi,\pi}|^p \right] \\ &\le \tilde{C} E\left[\int_s^T |\xi_r|^p \, dr \right]. \end{aligned}$$

Letting $s \to T$, we conclude (3.39) by the monotone convergence theorem, where it is used that $\xi \in L^p_{\mathcal{F}}(0, T; \mathbb{R})$ for any strategy (ξ, π) that has finite costs as η is bounded away from zero under assumption (A2). □

The following estimate is key to the verification argument. Together with the preceding lemma it allows us to show that $v(\cdot, \cdot)| \cdot |^p$ is indeed to value function associated with our control problem.

Lemma 3.4.4. *Under assumption (A1), for ever $(\xi,\pi) \in \bar{\mathcal{A}}(t,x)$ and $s \in [t,T)$ it holds,*

$$v(t,y)|x|^p \le E\left[v(s,Y_s^{t,y})|X_s^{\xi,\pi}|^p\right] + E\left[\int_t^s c(Y_r^{t,y},X_r^{\xi,\pi},\xi_r,\pi_r)\,dr\right].$$

Proof. Let us denote by $B(y,R)$ the open ball with radius $R>0$ centered at $y \in \mathbb{R}^d$ and introduce the first exit time

$$\tau_R = \inf\{r \ge t : Y_r^{t,y} \notin B(y,R)\}.$$

Since $v \in W_q^{1,2}((t,s)\times B(y,R))$ and $Y^{t,y}$ is non-degenerated, due to assumption (A1), Krylov's generalized Itô formula [Kry80, Theorem 2.10.1] applies to the stopped process $v(s\wedge\tau_R, Y_{s\wedge\tau_R}^{t,y})$. It yields that

$$\begin{aligned} v(t,y) = v(s\wedge\tau_R, Y_{s\wedge\tau_R}^{t,y}) + \int_t^{s\wedge\tau_R} \partial_t v(r,Y_r^{t,y}) + \mathcal{L}v(r,Y_r^{t,y})\,dr \\ - \int_t^{s\wedge\tau_R} \sigma(Y_r^{t,y})Dv(r,Y_r^{t,y})\,dW_r. \end{aligned}$$

This allows us to apply to $v(s\wedge\tau_R, Y_{s\wedge\tau_R}^{t,y})|X_{s\wedge\tau_R}^{\xi,\pi}|^p$ the classical integration by parts formula for semimartingales [JS03, Theorem 4.57] in order to obtain

$$\begin{aligned} v(t,y)|x|^p &= v(s\wedge\tau_R, Y_{s\wedge\tau_R}^{t,y})|X_{s\wedge\tau_R}^{\xi,\pi}|^p \\ &\quad - \int_t^{s\wedge\tau_R} \big\{\partial_t v(r,Y_r^{t,y})|X_r^{\xi,\pi}|^p + \mathcal{L}v(r,Y_r^{t,y})|X_r^{\xi,\pi}|^p \\ &\quad - p\xi_r v(r,Y_r^{t,y})\operatorname{sgn}(X_r^{\xi,\pi})|X_r^{\xi,\pi}|^{p-1} + \theta v(r,Y_r^{t,y})(|X_r^{t,x}-\pi_r|^p - |X_r^{t,x}|^p)\big\}\,dr \\ &\quad - \int_t^{s\wedge\tau_R} \sigma(Y_r^{t,y})Dv(r,Y_r^{t,y})|X_r^{\xi,\pi}|^p\,dW_r \\ &\quad - \int_t^{s\wedge\tau_R} v(r,Y_r^{t,y})(|X_{r-}^{\xi,\pi}-\pi_r|^p - |X_{r-}^{\xi,\pi}|^p)\,d\widetilde{N}_r, \end{aligned}$$

where $\widetilde{N}_r = N_r - \theta r$ denotes the compensated Poisson process. Both v and Dv are continuous and hence bounded on $[0,s]\times\overline{B}(y,R)$. Furthermore, $|X^{\xi,\pi}| \le |x|$ and $|\pi| \le |x|$, due to the monotonicity of the portfolio process. As a consequence, the above stochastic integrals are true martingales. Hence, recalling (3.14),

$$\begin{aligned} v(t,y)|x|^p &= E\left[v(s\wedge\tau_R, Y_{s\wedge\tau_R}^{t,y})|X_{s\wedge\tau_R}^{\xi,\pi}|^p\right] + E\left[\int_t^{s\wedge\tau_R} c(Y_r^{t,y},X_r^{\xi,\pi},\xi_r,\pi_r)\,dr\right] \\ &\quad - E\left[\int_t^{s\wedge\tau_R}\big\{(\partial_t+\mathcal{L})v(r,Y_r^{t,y})|X_r^{\xi,\pi}|^p + H(r,Y_r^{t,x},X_r^{\xi,\pi},\xi_r,\pi_r,v(\cdot,\cdot)|\cdot|)\big\}\,dr\right] \\ &\le E\left[v(s\wedge\tau_R, Y_{s\wedge\tau_R}^{t,y})|X_{s\wedge\tau_R}^{\xi,\pi}|^p\right] + E\left[\int_t^{s\wedge\tau_R} c(Y_r^{t,y},X_r^{\xi,\pi},\xi_r,\pi_r)\,dr\right] \\ &\quad - E\left[\int_t^{s\wedge\tau_R}\big\{(\partial_t+\mathcal{L})v(r,Y_r^{t,y}) + F(Y_r^{t,y},v(r,Y_r^{t,y}))\big\}\,|X_r^{\xi,\pi}|^p\,dr\right]. \end{aligned} \tag{3.43}$$

Since v satisfies (3.11) a.e., since $Y^{t,y}$ is non-degenerated, and because $|X^{\xi,\pi}| \leq |x|$, if follows from Krylov's estimate [Kry80, Theorem 2.4] that

$$E\left[\int_t^{s\wedge\tau_R} \left\{(\partial_t + \mathcal{L})v(r, Y_r^{t,y}) + F(Y_r^{t,y}, v(r, Y_r^{t,y}))\right\} |X_r^{\xi,\pi}|^p \, dr\right] = 0.$$

Hence,

$$v(t,y)|x|^p \leq E\left[v(s\wedge\tau_R, Y_{s\wedge\tau_R}^{t,y})|X_{s\wedge\tau_R}^{\xi,\pi}|^p\right] + E\left[\int_t^{s\wedge\tau_R} c(Y_r^{t,y}, X_r^{\xi,\pi}, \xi_r, \pi_r)\, dr\right].$$

Letting $R \to \infty$ the assertion follows from the polynomial growth condition on v and positivity of the cost function c. □

We are now ready to carry out the verification argument.

Proof of Proposition 3.1.9. Let $(\xi,\pi) \in \bar{\mathcal{A}}(t,x)$. By Lemma 3.4.4 and Lemma 3.4.3 letting $s \to T$ (assuming w.l.o.g. that (ξ,π) has finite costs) we get

$$v(t,y)|x|^p \leq J(t,y,x;\xi,\pi).$$

Finally note by Lemma 3.1.6 that equality holds in (3.43) if $\xi = \xi^*$ and $\pi = \pi^*$. Since v and c are both nonnegative this implies that

$$v(t,y)|x|^p \geq \lim_{s\to T} E\left[\int_t^s c(Y_r^{t,y}, X_r^{\xi,\pi}, \xi_r, \pi_r)\, dr\right] = J(t,y,x;\xi^*,\pi^*). \tag{3.44}$$

In particular (ξ^*,π^*) has finite costs. Hence, Lemma 3.4.3 applies to (ξ^*,π^*). Thus,

$$\begin{aligned} v(t,y)|x|^p &= E[v(s, Y_s^{t,y})|X_s^{\xi^*,\pi^*}|^p] + E\left[\int_t^s c(Y_r^{t,y}, X_r^{\xi^*,\pi^*}, \xi_r^*, \pi_r^*)\, dr\right] \\ &\longrightarrow J(t,y,x;\xi^*,\pi^*) \quad \text{as } s \to T. \end{aligned}$$

This shows that the strategy (ξ^*,π^*) is indeed optimal. □

3.5. Uniqueness in the non-Markovian framework

Within our Markovian framework we obtained optimal controls in feedback form. Of course, one may as well interpret the cost coefficients as processes η_t, γ_t and λ_t adapted to the filtration generated by the Brownian motion. This has been recently suggested by Ankirchner, Jeanblanc & Kruse [AJK14], which allowed them to analyze non-Markovian coefficients, while losing the feedback form of the optimal controls.

Disregarding in this section any passive orders and assuming the filtration to be solely generated by the Brownian motion the value function to the control problem consider in [AJK14] is given by

$$V_t(x) := \operatorname*{ess\,inf}_{\xi\in\mathcal{A}(t,x)} E\left[\int_t^T \eta_s|\xi_s|^p + \lambda_s|X_s^\xi|^p\, ds \,\middle|\, \mathcal{F}_t\right], \qquad (t,x) \in [0,T) \times \mathbb{R},$$

where $\xi \in L^0_{\mathcal{F}}(t,T;\mathbb{R})$ belongs to the set of admissible controls $\mathcal{A}(t,x)$ if the state process

$$X^\xi_s = x - \int_t^s \xi_r \, dr, \qquad t \le s \le T,$$

satisfies the liquidation constraint $X^\xi_T = 0$. In the non-Markovian framework the value function has been related by Peng [Pen92] (see also [GHQ15, HQZ16]) to the BSPDE:

$$-dV_t(x) = \inf_{\xi \in \mathbb{R}} \{-\xi \nabla V_t(x) + \eta_t |\xi|^p + \lambda_t |x|^p\} \, dt - \Psi_t(x) \, dW_t.$$

A solution to the BSPDE is a pair (V_t, Ψ_t) of adapted processes. The ansatz $V_t(x) = Y_t|x|^p$ and $\Psi_t(x) = Z_t|x|^p$ results in the BSDE:

$$-dY_t = \left\{ \lambda_t - \frac{|Y_t|^{\beta+1}}{\beta \eta_t^\beta} \right\} dt - Z_t \, dW_t, \quad 0 \le t < T; \quad \lim_{t \to T} Y_t = +\infty. \tag{3.45}$$

Under the assumptions $\eta \in L^2_{\mathcal{F}}(0,T;\mathbb{R}_+)$, $\eta^{-\beta} \in L^1_{\mathcal{F}}(0,T;\mathbb{R}_+)$, $\lambda \in L^2_{\mathcal{F}}(0,T^-;\mathbb{R}_+)$, and $E[\int_0^T (T-t)^p \lambda_t \, dt] < \infty$, existence of a minimal nonnegative solution

$$(Y,Z) \in L^2_{\mathcal{F}}(\Omega; C([0,T^-];\mathbb{R}_+)) \times L^2_{\mathcal{F}}(0,T^-;\mathbb{R}^n)$$

to (3.45) has been established in [AJK14]. We now show how our uniqueness argument can be applied to establish uniqueness in the BSDE setting when the coefficients are continuous. While the general shifting argument in the proof of our comparison principle fails in a non-time-homogeneous setting, it can be applied to establish the following a priori estimates.

Proposition 3.5.1. *For any nonnegative solution* (Y,Z) *to* (3.45) *the following estimates hold:*

$$\frac{1}{\sqrt[\beta]{E\left[\int_t^T \frac{1}{\eta_s^\beta} \, ds \,\middle|\, \mathcal{F}_t\right]}} \le Y_t \le \frac{1}{(T-t)^p} E\left[\int_t^T \eta_s + (T-s)^p \lambda_s \, ds \,\middle|\, \mathcal{F}_t\right] =: \overline{Y_t}. \tag{3.46}$$

Proof. The lower estimate has already been established in [AJK14] for the minimal nonnegative solution. To establish the upper estimate first note that $\overline{Y} = (\overline{Y}_t)_{t \in [0,T)}$ is a supersolution to (3.45). Yet, one can not directly compare $\overline{Y}$ and Y at the terminal time. As a workaround, similarly as in [Pop06], we modify Pardoux's proof of his comparison principle [Par99, Theorem 2.4] for BSDEs with monotone drivers by shifting the singularity of $\overline{Y}$. That is, for $\delta > 0$ we define $(\overline{Y_t}^\delta)_{t \in [0,T-\delta)}$ by

$$\overline{Y_t}^\delta = \frac{1}{(T-\delta-t)^p} E\left[\int_t^{T-\delta} \eta_s + (T-\delta-s)^p \lambda_s \, ds \,\middle|\, \mathcal{F}_t\right].$$

These processes are again supersolutions to (3.45) but with the singularity at $t = T-\delta$. Precisely, it holds that

$$-d\overline{Y_t}^\delta = \underbrace{\lambda_t + \frac{\eta_t}{(T-\delta-t)^p} - \frac{p\overline{Y_t}^\delta}{T-\delta-t}}_{=:g^\delta(t,\overline{Y_t}^\delta)}\,dt - \overline{Z_t}^\delta\,dW_t, \qquad 0 \le t < T-\delta,$$

for some $\overline{Z}^\delta \in \bigcap_{t\in[0,T-\delta)} L^2_{\mathcal{F}}(0,t;\mathbb{R}^n)$ given by the Martingale Representation Theorem with the singular terminal value

$$\lim_{t\to T-\delta} \overline{Y_t}^\delta = +\infty.$$

A calculation as in (3.25) verifies that for all $0 \le t < T-\delta$ and $y \in \mathbb{R}$,

$$g^\delta(t,y) \ge \lambda_t - \frac{|y|^{\beta+1}}{\beta\eta_t^\beta} =: f(t,y).$$

We now consider the difference of Y and $\overline{Y}^\delta$ for $0 \le t \le s < T-\delta$:

$$\begin{aligned}\overline{Y_t}^\delta - Y_t &= E\left[\overline{Y_s}^\delta - Y_s + \int_t^s g^\delta(r,\overline{Y_r}^\delta)\,dr - \int_t^s f(r,Y_r)\,dr \,\middle|\, \mathcal{F}_t\right]\\ &= E\left[\overline{Y_s}^\delta - Y_s - \int_t^s \frac{p(\overline{Y_r}^\delta - Y_r)}{T-\delta-r}\,dr + \int_t^s g^\delta(r,Y_r) - f(r,Y_r)\,dr \,\middle|\, \mathcal{F}_t\right].\end{aligned}$$

By the solution formula for linear BSDEs:

$$\overline{Y_t}^\delta - Y_t = E\left[(\overline{Y_s}^\delta - Y_s)e^{-\int_t^s \frac{p}{T-\delta-r}\,dr} + \int_t^s g^\delta(r,Y_r) - f(r,Y_r)\,dr \,\middle|\, \mathcal{F}_t\right].$$

Therefore,

$$\overline{Y_t}^\delta - Y_t \ge E\left[\left(\overline{Y_s}^\delta - \sup_{t\le s\le T-\delta} Y_s\right)e^{-\int_t^s \frac{p}{T-\delta-r}\,dr} \,\middle|\, \mathcal{F}_t\right].$$

Now, letting $s \to T-\delta$ this yields $\overline{Y_t}^\delta - Y_t \ge 0$ by Fatou's lemma. This completes the proof since

$$\overline{Y_t} - Y_t = \lim_{\delta\to 0} \overline{Y_t}^\delta - Y_t$$

by the monotone convergence theorem. □

The upper estimate in (3.46) may be used to establish the analogue to Lemma 3.4.3. Here we need to impose the following essentially boundedness assumption on the coefficients:

(A4) $\eta, \eta^{-1}, (T-\cdot)^p\lambda \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}_+)$.

Corollary 3.5.2. *Under assumption (A4), for any solution*

$$(Y, Z) \in L^2_{\mathcal{F}}(\Omega; C([0, T^-]; \mathbb{R}_+)) \times L^2_{\mathcal{F}}(0, T^-; \mathbb{R}^n)$$

to (3.45) *and any admissible control* $\xi \in \mathcal{A}(t, x)$ *with finite costs it holds that*

$$\lim_{s \to T} E\left[Y_s |X^\xi_s|^p \,\middle|\, \mathcal{F}_t \right] = 0. \tag{3.47}$$

Proof. From the upper estimate in (3.46) and the tower property,

$$E\left[Y_s |X^\xi_s|^p \middle| \mathcal{F}_t \right] \leq E\left[\frac{\int_s^T \{\eta_r + (T-r)^p \lambda_r\}\, dr}{(T-t)^p} |X^\xi_s|^p \middle| \mathcal{F}_t \right].$$

Taking the liquidation constraint into account, we obtain after an application of Jensen's inequality,

$$\begin{aligned} E\left[Y_s |X^\xi_s|^p \middle| \mathcal{F}_t \right] &\leq E\left[\frac{\int_s^T \{\eta_r + (T-r)^p \lambda_r\}\, dr}{T-t} \int_s^T |\xi_r|^p\, dr \middle| \mathcal{F}_t \right] \\ &\leq CE\left[\int_s^T |\xi_r|^p\, dr \middle| \mathcal{F}_t \right]. \end{aligned}$$

Hence, letting $s \to T$, we conclude (3.47) by the monotone convergence theorem, where it is used that $\xi \in L^p_{\mathcal{F}}(0, T; \mathbb{R})$ for any control ξ that has finite costs as η is bounded away from zero under assumption (A4). □

Finally, by a verification argument analogous to the one in Section 3.4 we obtain $V_t(x) = Y_t|x|^p$ for any nonnegative solution $(Y, Z) \in L^2_{\mathcal{F}}(\Omega; C([0, T^-]; \mathbb{R}_+)) \times L^2_{\mathcal{F}}(0, T^-; \mathbb{R}^n)$ and conclude:

Theorem 3.5.3. *Under assumption (A4) uniqueness holds for problem* (3.45) *in the class of nonnegative solutions in* $L^2_{\mathcal{F}}(\Omega; C([0, T^-]; \mathbb{R}_+)) \times L^2_{\mathcal{F}}(0, T^-; \mathbb{R}^n)$.

Proof. We may again restrict the argument without loss of generality [AJK14, Lemma 1.6] to monotone controls $\xi \in \mathcal{A}(t, x)$ with finite costs. By Itô-Kunita formula [Kun84, Theorem I.8.1], since $Y_t|x|^p$ solves the stochastic HJB equation,

$$Y_t|x|^p \leq E\left[Y_s|X^\xi_s|^p + \int_t^s \eta_r |\xi_r|^p + \lambda_r |X^\xi_r|^p\, dr \middle| \mathcal{F}_t \right], \qquad t \leq s < T.$$

Letting $s \to T$, we conclude $Y_t|x|^p \leq V_t(x)$ by Corollary 3.5.2. Thus, $Y_t \leq V_t(1)$. But $V_t(1)$ is characterized in [AJK14] as the minimal nonnegative solution to (3.45). Hence, $Y_t = V_t(1)$ is unique. □

3.6. Conclusion

In this chapter we proposed a novel approach to establishing smooth solutions to stochastic optimal control problems with singular terminal state constraints in a Markovian framework. Under standard assumptions on the diffusion and cost coefficients we proved that there exists at most one continuous viscosity and hence strong/classical solution to the HJB equation. As a byproduct we obtained a uniqueness theorem in a non-Markovian framework that complements results in [AJK14]. Our main contribution is the existence of a classical solution under boundedness and differentiability assumptions on the cost function. Existence of a viscosity solution is still open. In its present form our comparison principle only applies to continuous sub- and supersolutions, and hence does not allow us to apply Perron's method to establish the existence of a viscosity solution. Our verification argument uses Krylov's generalized Itô formula. As such it applies to strong and classical solutions. It is not hard, though, to extend the verification argument to viscosity solutions.

4. Optimal Trade Execution with Instantaneous Price Impact and Stochastic Resilience

4.1. Introduction and overview

Let $T \in (0, \infty)$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be filtered probability space that carries a m-dimensional standard Brownian motion $W = (W_t)_{t\in[0,T]}$. We assume throughout that $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by W completed by all the null sets and that $\mathcal{F} = \mathcal{F}_T$. We denote by $L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^d)$ and $L^\infty_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^d))$, respectively, the set of progressively measurable $\mathbb{R}^d$-valued, respectively, continuous processes that are essentially bounded. $L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$ denotes the set of progressively measurable $\mathbb{R}^d$-valued processes $(Y_t)_{t\in[0,T]}$ such that $E[\int_0^T |Y_t|^2 \, dt] < \infty$, and $L^2_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^d))$ denotes the subset of all such processes with continuous sample paths such that $E[\sup_{t\in[0,T]} |Y_t|^2] < \infty$. All equations and inequalities are to be understood in the $\mathbb{P}$-a.s. sense.

In this chapter we address the linear-quadratic non-Markovian stochastic control problem

$$\operatorname*{ess\,inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E}\left[\int_0^T \tfrac{1}{2}\eta\xi_s^2 + \xi_s Y_s + \tfrac{1}{2}\lambda_s X_s^2 \, ds\right] \tag{4.1}$$

subject to

$$\begin{cases} X_t = x - \displaystyle\int_0^t \xi_s \, ds, \quad t \in [0, T], \\ X_T = 0, \\ Y_t = y + \displaystyle\int_0^t \{-\rho_s Y_s + \gamma\xi_s\} \, ds, \quad t \in [0, T]. \end{cases} \tag{4.2}$$

Here, η and γ are positive constants and ρ and λ are progressively measurable, non-negative and essentially bounded stochastic processes:

$$\eta > 0, \gamma \in \mathbb{R}_+; \quad \rho, \lambda \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}_+).$$

We sometimes write $X^{t,x}_s$ and $Y^{t,y}_s$ for $0 \le t \le s \le T$ to indicate that the state process starts at time $t \in [0, T]$ in $(x, y) \in \mathbb{R} \times \mathbb{R}$. For the given initial data $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ we denote by $\mathcal{A}(t, x)$ the set of all *admissible controls*, that is the class of all controls $\xi \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$ such that the terminal state constraint $X_T = 0$ is satisfied a.s.

Control problems of the above form arise in models of optimal portfolio liquidation under market impact with stochastic resilience. In such models X_t denotes the portfolio an investor holds at time $t \in [0, T]$, ξ_t denotes the rate at which the stock is traded at time $t \in [0,T]$, and the terminal state constraint $X_T = 0$ is the *liquidation constraint*. The process Y describes the *persistent price impact* caused by past trades in a block-shaped limit order book market with constant order book depth $1/\gamma > 0$ as in Obizhaeva and Wang [OW13]. One interpretation is that the trading rate ξ adds a drift to an underlying fundamental martingale price price process. The stochastic process $\rho \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}_+)$ describes the rates at which the order book recovers from past trades. The constant $\eta > 0$ describes an additional *instantaneous impact factor* as in Almgren and Chriss [AC01]. The first two terms of running cost term in (4.1) capture the cost resulting from the instantaneous and the persistent impact, respectively. The third term can be interpreted as a measure of the market risk associated with an open position. It penalizes slow liquidation. We allow the risk factor λ to be stochastic.

The remainder of this chapter is structured as follows. The stochastic control problem is formulated in Section 4.2. The a priori estimates and asymptotic behavior of the solution is established in Section 4.3. Existence to the HJB equation is proven in Section 4.4. The verification argument is carried out in Section 4.5. In Appendix A.3 we recall the multi-dimensional comparison principle for BSDEs and formulate a local L^∞-existence result for BSDEs with locally Lipschitz drivers.

Notational convention. Whenever the notation T^- appears we mean that the statement holds for all the $T' < T$ when T^- is replaced by T', e.g.,

$$L^2_{\mathcal{F}}(0,T^-;\mathbb{R}^{d\times m}) = \bigcap_{T'<T} L^2_{\mathcal{F}}(0,T';\mathbb{R}^{d\times m}).$$

Furthermore, for $Y \in L^\infty_{\mathcal{F}}(\Omega, C([0,T^-];\mathbb{R}))$ we mean by $L^\infty\text{-}\lim_{t\to T} Y_t = \infty$ that for every $C > 0$ there exists $\tau < T$ such that $Y_t \geq C$ for all $t \in [\tau, T)$, $\mathbb{P}$-a.e.

4.2. Main result

For any initial state $(t,x,y) \in [0,T) \times \mathbb{R} \times \mathbb{R}$ we define by

$$V_t(x,y) := \operatorname*{ess\,inf}_{\xi \in \mathcal{A}(t,x)} \mathbb{E}\left[\int_t^T \tfrac{1}{2}\eta\xi_s^2 + \xi_s Y_s + \tfrac{1}{2}\lambda_s X_s^2\, ds \,\middle|\, \mathcal{F}_t\right] \tag{4.3}$$

the *value function* of the stochastic control problem (4.1) subject the state dynamics (4.2), where

$$\eta > 0, \gamma \in \mathbb{R}_+; \quad \rho, \lambda \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}_+),$$

and $\mathcal{A}(t,x)$ denotes the class of all strategies $\xi \in L^2_{\mathcal{F}}(t,T;\mathbb{R})$ that satisfy the terminal state constraint

$$X_T = 0 \quad \text{a.s.}$$

Remark 4.2.1. Notice that $X, Y \in L^2_{\mathcal{F}}(\Omega; C([t,T];\mathbb{R}))$ for any admissible strategy as $\xi \in L^2_{\mathcal{F}}(t,T;\mathbb{R})$ and $\rho \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}_+)$.

We solve the control problem by solving the corresponding stochastic Hamilton-Jacobi-Bellman (HJB) equation. Stochastic HJB equations for non-Markovian control problems were first introduced by Peng [Pen92]. In our model the stochastic HJB equation is given by the first-order stochastic partial differential equation,

$$\begin{aligned} -dV_t(x,y) = \inf_{\xi\in\mathbb{R}}\{-\xi\partial_x V_t(x,y) - (\rho_t y - \gamma\xi)\partial_y V_t(x,y) + \tfrac{1}{2}\eta\xi^2 + \xi y + \tfrac{1}{2}\lambda_t x^2\}\,dt \\ - Z_t(x,y)\,dW_t. \end{aligned} \tag{4.4}$$

A pair of random fields $(V,Z) : \Omega \times [0,T) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^m$ is called a *classical solution* to the above equation if it satisfies the following conditions:

– for each $t \in [0,T)$, $V_t(x,y)$ is continuously differentiable in x and y,

– for $(x,y) \in \mathbb{R}^2$, $(V_t(x,y), \partial_x V_t(x,y), \partial_y V_t(x,y))_{t\in[0,T)} \in L^\infty_{\mathcal{F}}(\Omega; C([0,T^-];\mathbb{R}^3))$,

– for each $(x,y) \in \mathbb{R}^2$, $(Z_t(x,y))_{t\in[0,T)} \in L^2_{\mathcal{F}}(0,T^-;\mathbb{R}^m)$,

– for all $0 \le t \le s < T$ and $x,y \in \mathbb{R}$ it holds that

$$\begin{aligned} V_t(x,y) = V_s(x,y) + \int_t^s \inf_{\xi\in\mathbb{R}}\{-\xi\partial_x V_r(x,y) - (\rho_r y - \gamma\xi)\partial_y V_r(x,y) \\ + \tfrac{1}{2}\eta\xi^2 + \xi y + \tfrac{1}{2}\lambda_r x^2\}\,dr - \int_t^s Z_r(x,y)\,dW_r. \end{aligned}$$

We prove the existence of a unique classical solution to the equation (4.4) and show that the value function is given by the random field V. The linear-quadratic structure of the control problem suggest the ansatz

$$\begin{aligned} V_t(x,y) &= \tfrac{1}{2}A_t x^2 + B_t xy + \tfrac{1}{2}C_t y^2 \\ Z_t(x,y) &= \tfrac{1}{2}Z^A_t x^2 + Z^B_t xy + \tfrac{1}{2}Z^C_t y^2 \end{aligned} \tag{4.5}$$

for the solution $(V(x,y), Z(x,y))$ to the HJB equation. The following lemma shows that this ansatz reduces our HJB equation to the following three-dimensional stochastic Riccati equation:

$$\begin{cases} -dA_t = \left\{\lambda_t - \eta^{-1}(A_t - \gamma B_t)^2\right\}dt - Z^A_t\,dW_t \\ -dB_t = \left\{-\rho_t B_t + \eta^{-1}(\gamma C_t - B_t + 1)(A_t - \gamma B_t)\right\}dt - Z^B_t\,dW_t \\ -dC_t = \left\{-2\rho_t C_t - \eta^{-1}(\gamma C_t - B_t + 1)^2\right\}dt - Z^C_t\,dW_t. \end{cases} \tag{4.6}$$

Lemma 4.2.2. *If the vector*

$$\big((A,B,C),(Z^A,Z^B,Z^C)\big) \in L^\infty_{\mathcal{F}}(\Omega; C([0,T^-];\mathbb{R}^3)) \times L^2_{\mathcal{F}}(0,T^-;\mathbb{R}^{3\times m})$$

solves the BSDE system (4.6)*, then the random field* (V,Z) *given by the linear-quadratic ansatz* (4.5) *solves the HJB equation* (4.4) *such that the infimum in* (4.4) *is attained by*

$$\xi^*_t(x,y) = \eta^{-1}(A_t - \gamma B_t)x - \eta^{-1}(\gamma C_t - B_t + 1)y. \tag{4.7}$$

Proof. Let us fix $(t, x, y) \in [0, T) \times \mathbb{R} \times \mathbb{R}$. The Hamiltonian

$$\begin{aligned} h(\xi) &= -\xi \partial_x V_t(x,y) - (\rho_t y - \gamma\xi)\partial_y V_t(t,x) + \tfrac{1}{2}\eta\xi^2 + \xi y + \tfrac{1}{2}\lambda_t x^2 \\ &= \tfrac{1}{2}\eta^{-1}\left(\eta\xi - \partial_x V_t(x,y) + \gamma\partial_y V_t(t,x) + y\right)^2 \\ &\quad - \tfrac{1}{2}\eta^{-1}\left(\partial_x V_t(x,y) - \gamma\partial_y V_t(x,y) - y\right)^2 - \rho_t y \partial_y V_t(x,y) + \tfrac{1}{2}\lambda_t x^2 \end{aligned}$$

is minimized at

$$\xi^* = \eta^{-1}(\partial_x V_t(x,y) - \gamma\partial_y V_t(t,x) - y).$$

In terms of the linear-quadratic ansatz (4.5), we obtain (4.7) and

$$\begin{aligned} h(\xi^*) &= -\tfrac{1}{2}\eta^{-1}((A_t - \gamma B_t)x - (\gamma C_t - B_t + 1)y)^2 - \rho_t y(B_t x + C_t y) + \tfrac{1}{2}\lambda_t x^2 \\ &= \tfrac{1}{2}(\lambda_t - \eta^{-1}(A_t - \gamma B_t)^2)x^2 + (-\rho_t B_t + \eta^{-1}(A_t - \gamma B_t)(\gamma C_t - B_t + 1))xy \\ &\quad + \tfrac{1}{2}(-2\rho_t C_t - \eta^{-1}(\gamma C_t - B_t + 1)^2)y^2. \end{aligned}$$

□

In order to guarantee the uniqueness of a solution to the HJB equation we need to impose a suitable terminal condition. Due to the terminal state constraint $X_T = 0$ we expect the trading rate ξ to tend to infinity for any non-trivial initial position as $t \to T$. We further expect the resulting trading cost to dominate any resilience effect. As a result, we expect that

$$V_t(x,y) \sim V_t^{\rho=0}(x,y) \quad \text{as } t \to T$$

where $V_t^{\rho=0}(x,y)$ denotes the value function corresponding to the control problem with $\rho \equiv 0$. If $\rho \equiv 0$, then $Y^{\rho=0} = y + \gamma(x - X)$ and

$$\int_t^T \xi_s Y_s^{\rho=0}\, ds = xy + \tfrac{1}{2}\gamma x^2,$$

independently of the strategy $\xi \in \mathcal{A}(t,x)$. Hence,

$$\begin{aligned} V_t^{\rho=0}(x,y) &= \operatorname*{ess\,inf}_{\xi\in\mathcal{A}(t,x)} \mathbb{E}\left[\int_t^T \tfrac{1}{2}\eta\xi_s^2 + \tfrac{1}{2}\lambda_s X_s^2\, ds \,\middle|\, \mathcal{F}_t\right] + xy + \tfrac{1}{2}\gamma x^2 \\ &= \tfrac{1}{2}(\tilde{A}_t + \gamma)x^2 + xy, \end{aligned}$$

where $\tilde{A}$ is characterized in [AJK14, GHS16] as the unique solution to the BSDE with singular terminal value

$$\begin{cases} -d\tilde{A}_t = \lambda_t - \eta^{-1}\tilde{A}_t^2\, dt - Z_t\, dW_t \\ \lim\limits_{t\to\infty} \tilde{A}_t = \infty \text{ in } L^\infty. \end{cases}$$

We therefore expect the coefficients of the linear-quadratic ansatz (4.5) to satisfy

$$(A_t, B_t, C_t) \longrightarrow (\infty, 1, 0) \quad \text{in } L^\infty \text{ as } t \to T. \tag{4.8}$$

The next theorem establishes an existence of solutions result for the BSDE system (4.6) when imposed with the terminal state constraint (4.8). The proof is given in Section 4.4. It is based on a multi-dimensional generalization of the asymptotic expansion approached introduced in [GHS16].

Theorem 4.2.3. *The BSDE system* (4.6) *imposed with the singular terminal condition* (4.8) *admits at least one solution*

$$((A,B,C),(Z^A,Z^B,Z^C)) \in L^\infty_{\mathcal{F}}(\Omega;C([0,T^-];\mathbb{R}^3)) \times L^2_{\mathcal{F}}(0,T^-;\mathbb{R}^{3\times m}).$$

The next theorem verifies the preceding heuristics; its proof is given in Section 4.5. In particular, it states that the value function is indeed of the form (4.5). As a result, there exists at most solution to the BSDE system (4.6) that satisies (4.8).

Theorem 4.2.4. *Let*

$$((A,B,C),(Z^A,Z^B,Z^C)) \in L^\infty_{\mathcal{F}}(\Omega;C([0,T^-];\mathbb{R}^3)) \times L^2_{\mathcal{F}}(0,T^-;\mathbb{R}^{3\times m})$$

be a solution to the BSDE system (4.6) *that satisfies the singular terminal state constraint* (4.8). *Then, the value function is of the linear-quadratic form* (4.5) *and the optimal control is given by the feedback form* (4.7). *In particular, the system admits at most one solution that satisfies* (4.8).

Example 4.2.5. In a deterministic benchmark model with a risk neutral investor ($\lambda \equiv 0$) and constant deterministic resilience ($\rho_t \equiv \rho > 0$) the above BSDE system reduces to the following ODE system:

$$\begin{cases} -\dot{A}_t = -\eta^{-1}(A_t - \gamma B_t)^2, & 0 \le t < T; \qquad \lim_{t\to T} A_t = +\infty; \\ -\dot{B}_t = -\rho B_t + \eta^{-1}(\gamma C_t - B_t + 1)(A_t - \gamma B_t), & 0 \le t < T; \qquad \lim_{t\to T} B_t = 1; \\ -\dot{C}_t = -2\rho C_t - \eta^{-1}(\gamma C_t - B_t)^2, & 0 \le t < T; \qquad \lim_{t\to T} C_t = 0. \end{cases}$$

The optimal trading strategies for different choices of the instantaneous impact factor η and the optimal trading strategies of the benchmark models by Almgren and Chriss [AC01] and Obizhaeva and Wang [OW13] are depicted in Figure 4.1 below. The optimal trading strategy resembles that of the Almgren–Chriss model for large instantaneous impact factors while it resembles that of the Obizhaeva–Wang model with singular controls for small instantaneous impact factors. This suggests that our model can be viewed as a blend of the two extreme cases with only instantaneous, respectively only permanent market impact.

4.3. A Priori Estimates

In this section we establish a priori estimates for the BSDE system (4.6). The estimates will be key for both, the proof of the existence of solutions and the verification theorem. Throughout, let

$$((A,B,C),(Z^A,Z^B,Z^C)) \in L^\infty_{\mathcal{F}}(\Omega;C([0,T^-];\mathbb{R}^3)) \times L^2_{\mathcal{F}}(0,T^-;\mathbb{R}^{3\times m})$$

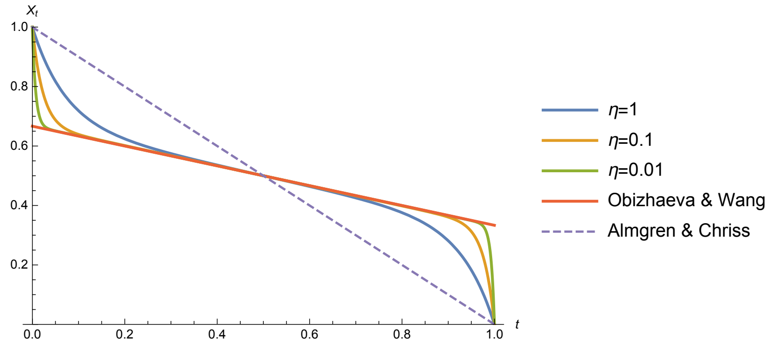

Figure 4.1.: The optimal trading strategy in a deterministic benchmark model for different instantaneous impact factors η compared to the models by Almgren and Chriss [AC01] and by Obizhaeva and Wang [OW13]. Here, $x = 1$, $y = 0$, $\lambda \equiv 0$, $\gamma = 100$, $T = 1$, and $\rho \equiv 1$.

denote any solution to (4.6) that satisfies (4.8). It will be convenient to also consider the processes

$$D := \eta^{-1}(A - \gamma B) \qquad \text{and} \qquad E := \eta^{-1}(\gamma C - B + 1)$$

that appear in the feedback form (4.7) of the candidate optimal control. The equations for D and E read:

$$-dD_t = \{\eta^{-1}\lambda_t - D_t^2 + \eta^{-1}\gamma\rho_t B_t - \gamma E_t D_t\}\, dt - Z_t^D\, dW_t$$

and

$$\begin{aligned} -dE_t &= \{2\eta^{-1}\rho_t - 2\rho_t E_t - \gamma E_t^2 + \eta^{-1}\rho_t B_t - E_t D_t\}\, dt - Z_t^E\, dW_t \\ &= \{\eta^{-1}\rho_t(1 - \gamma C_t) - \rho_t E_t - \gamma E_t^2 - E_t D_t\}\, dt - Z_t^E\, dW_t. \end{aligned}$$

In order to establish the a priori estimates we first determine the range of the processes $A, \dots, E$. The proof of the following lemma uses the multi-dimensional comparison principle for BSDEs, due to Hu and Peng [HP06] presented in the Appendix.

Lemma 4.3.1. *It holds that $A, D \geq 0$ and $B, -\gamma C, \eta E \in [0, 1]$, $d\mathbb{P} \times dt$-a.e.*

Proof. We first note that $B, C \in L^\infty_{\mathcal{F}}(\Omega; C([0, T^-]; \mathbb{R}))$ together with the L^∞-convergence of B_t and C_t as $t \to T$ implies $B, C \in L^\infty_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}))$ and hence $E \in L^\infty_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}))$.

The nonpositivity of C follows from the solution formula for linear BSDEs with essentially bounded coefficients [PR14, Proposition 5.31]. Indeed, from

$$\begin{cases} -dC_t = \{-2\rho_t C_t - \eta E_t^2\}\, dt - Z_t^C\, dW_t, \quad 0 \leq t < T, \\ \quad C_T = 0, \end{cases}$$

we obtain that

$$C_t = -\mathbb{E}\left[\int_t^T \eta E_s^2 e^{-\int_t^s 2\rho_r\,dr}\,ds \middle| \mathcal{F}_t\right] \le 0. \tag{4.9}$$

The non-negativity of E follows from similar arguments. In fact,

$$-dE_t = \{-(\rho_t + \gamma E_t + D_t)E_t + \eta^{-1}\rho_t(1-\gamma C_t)\}dt - Z_t^E\,dW_t, \quad 0 \le t < T.$$

Even though D_t is singular at $t = T$, we may apply the solution formula on $[0,\tau]$ for all $\tau < T$. This yields,

$$E_t = \mathbb{E}\left[E_\tau e^{-\int_t^\tau (\rho_r+\gamma E_r + D_r)\,dr} + \int_t^\tau \eta^{-1}\rho_s(1-\eta C_s)e^{-\int_t^s(\rho_r+\gamma E_r+D_r)\,dr}\,ds \middle| \mathcal{F}_t\right]. \tag{4.10}$$

The L^∞-convergence of D_t to ∞ as $t \to T$ together with the fact that $D \in L^\infty_{\mathcal{F}}(\Omega; C([0,T^-];\mathbb{R}))$ implies that D is essentially bounded below on $[0,T]$. Since ρ, C and E are essentially bounded we can apply the dominated convergence theorem to interchange the limit and the expectation in (4.10) when letting $\tau \to T$ in (4.10). As a result, $E \ge 0$ because $C \le 0$ and because

$$E_\tau \to E_T = \eta^{-1}(\gamma C_T - B_T + 1) = 0 \quad \text{as } \tau \to T.$$

In order to prove that $B, D \ge 0$ we need we need to consider their *joint* dynamics. First, due to the (improper) L^∞-convergence of B_t and D_t as $t \to T$ there exists a deterministic time $\tau < T$ such that $B, D \ge 0$ on $[\tau, T]$. Let us consider the BSDE system for B and D on $[0,\tau]$:

$$\begin{cases} -dB_t = \{-\rho_t B_t + \eta E_t D_t\}\,dt - Z_t^B\,dW_t \\ -dD_t = \left\{\eta^{-1}\lambda_t - D_t^2 + \eta^{-1}\gamma\rho_t B_t - \gamma E_t D_t\right\}dt - Z_t^D\,dW_t. \end{cases}$$

Since ρ, E, and D are essentially bounded on $[0,\tau]$ we may assume without loss of generality by a standard truncation argument in the D-variable that this system is $d\mathbb{P}\times dt$-a.e. uniformly Lipschitz continuous in B and D. Furthermore, the system is quasi-monotone because $E, \rho \ge 0$. Hence, we may apply the comparison theorem for multi-dimensional BSDEs given in Proposition A.3.1 in the Appendix with

$$\begin{aligned} f^1(t,B,D) &= (-\rho_t B + \eta E_t D, -D^2 + \eta^{-1}\gamma\rho_t B - \gamma E_t D) \\ f^2(t,B,D) &= (-\rho_t B + \eta E_t D, \eta^{-1}\lambda_t - D^2 + \eta^{-1}\gamma\rho_t B - \gamma E_t D) \end{aligned}$$

(up to truncation in D) and terminal conditions $Y_\tau^1 = (0,0)$ and $Y_\tau^2 = (B_\tau, D_\tau) \ge Y_\tau^1$, respectively. As the unique solution to the first BSDE system satisfies $Y_t^1 \equiv (0,0)$, we see that $(B_t, D_t) = Y_t^2 \ge (0,0)$ for all $t \in [0,\tau]$. Hence the process (B,D) is non-negative.

Finally, we conclude from $B, -\gamma C, \eta E \ge 0$ and $\eta E = \gamma C - B + 1$ that

$$B, -\gamma C, \eta E \le 1. \qquad \square$$

We are now ready to establish the a priori estimates.

Proposition 4.3.2. *In terms of* $\kappa := \sqrt{2\eta^{-1}\max\{\|\lambda\|_{L^\infty}, \gamma\|\rho\|_{L^\infty}\}}$ *the following a priori estimates hold* $d\mathbb{P}\times dt$*-a.e.:*

$$\begin{aligned} \underline{D}_t := \frac{\eta^{-1}\gamma}{e^{\eta^{-1}\gamma(T-t)}-1} &\le D_t \le \kappa\coth\left(\kappa(T-t)\right) =: \overline{D}_t, \\ \underline{B}_t := e^{-\|\rho\|_{L^\infty}(T-t)} &\le B_t \le 1, \\ 0 &\le E_t \le \gamma^{-1}\kappa\tanh(\kappa(T-t)) =: \overline{E}_t \end{aligned}$$

Proof. Since $D\ge 0$ we may write the BSDE for D in monotone form. That is,

$$-dD_t = \left\{\eta^{-1}\lambda_t - |D_t|D_t + \eta^{-1}\gamma\rho_t B_t - \gamma E_t|D_t|\right\}dt - Z_t^D\,dW_t.$$

The lower and upper estimate for D solve

$$-d\underline{D}_t = \{-|\underline{D}_t|\underline{D}_t - \eta^{-1}\gamma|\underline{D}_t|\}\,dt,$$

and

$$-d\overline{D}_t = \{\kappa^2 - |\overline{D}_t|\overline{D}_t\}\,dt,$$

respectively. The preceding equations are time-homogeneous. Thus, for any $\delta>0$ the processes $\underline{D}^\delta_t := \underline{D}_{t-\delta}$ and $\overline{D}^\delta_t := \overline{D}_{t+\delta}$ still satisfy the respective equations but with singularities at $t=T+\delta$ and $t=T-\delta$, respectively. Since D is essentially bounded on $[0,T-\delta]$ and $\lim_{t\to T-\delta}\overline{D}^\delta_t = \infty$ in L^∞ there exits $s\in[0,T-\delta]$ such that $D\le\overline{D}^\delta$ on $[s,T-\delta)$. Because $B\le 1$ and $-ED\le 0$, we have for all $(t,y)\in[0,s]\times\mathbb{R}$,

$$\eta^{-1}\lambda_t - |y|y + \eta^{-1}\gamma\rho_t B_t - \gamma E_t D_t \le \kappa^2 - |y|y.$$

Hence, the classical one-dimensional comparison theorem for BSDEs with monotone drivers [PR14, Proposition 5.33] yields $D\le\overline{D}^\delta$ on $[0,s]$. Finally, letting $\delta\to 0$ yields $D\le\overline{D}$ on $[0,T)$ by the continuity of $\overline{D}$.

In order to establish $\underline{D}\le D$ on $[0,T)$ one argues similarly. In this case the comparison argument is justified by the inequality

$$-|y|y - \eta^{-1}\gamma|y| \le \lambda_t - |y|y + \eta^{-1}\gamma\rho_t B_t - \gamma E_t|y|.$$

Next, we establish the upper estimate for E. Since $E,D\ge 0$ we may again assume that the BSDE for E is monotone, that is

$$-dE_t = \{2\eta^{-1}\rho_t - 2\rho_t E_t - \gamma|E_t|E_t - \eta^{-1}\rho_t B_t - E_t D_t\}\,dr - Z_t^E\,dW_t.$$

Since $E,B,D\ge 0$ we have for all $(t,y)\in[0,T)\times\mathbb{R}$ that

$$2\eta^{-1}\rho_t - 2\rho_t E_t - \gamma|y|y - \eta^{-1}\gamma\rho_t B_t - E_t D_t \le \gamma^{-1}\kappa^2 - \gamma|y|y. \tag{4.11}$$

Let us consider for $\delta > 0$ the deterministic process

$$\overline{E}^{\delta}_t = \gamma^{-1}\kappa \tanh\left(\kappa(T-\delta-t) + \operatorname{arc\,tanh}(\gamma\kappa^{-1}\|E_{T-\delta}\|_{L^\infty})\right), \qquad 0 \le t \le T-\delta.$$

Then,

$$\begin{cases} -d\overline{E}^{\delta}_t = \gamma^{-1}\kappa^2 - \gamma|\overline{E}^{\delta}_t|\overline{E}^{\delta}_t, & 0 \le t \le T-\delta \\ \overline{E}^{\delta}_{T-\delta} = \|E_{T-\delta}\|_{L^\infty}. \end{cases}$$

Hence, recalling (4.11), the one-dimensional comparison theorem implies

$$E_t \le \overline{E}^{\delta}_t, \qquad t \in [0, T-\delta].$$

Since $\|E_T\|_{L^\infty} = 0$, letting $\delta \to 0$ completes the proof.

Finally, to establish the lower estimate for B one notices that $\underline{B}$ solves

$$-d\underline{B}_t = -\|\rho\|_{L^\infty}\underline{B}_t\,dt, \quad 0 \le t \le T; \qquad \underline{B}_T = 1,$$

and is hence a subsolution to the BSDE for B. At this point, we already now that the potential singular term E_tD_t in the BSDE for B behaves well (being bounded by $\overline{E}_t\overline{D}_t = \gamma^{-1}\kappa^2$) on the entire interval $[0,T]$. Hence, no shifting argument at the terminal time is needed in this step and we conclude directly by comparison that $\underline{B} \le B$. □

From the a priori estimates we obtain the asymptotic behavior of our BSDE system at the terminal time as stated in the following corollary. The asymptotic at the terminal time is key to our existence result.

Corollary 4.3.3. *The following asymptotic behaviors hold in L^∞ as $t \to T$:*

$$\begin{aligned} (T-t)A_t &= \eta + O(T-t), \\ B_t &= 1 + O(T-t), \\ C_t &= O((T-t)^3). \end{aligned}$$

Proof. The asymptotic behavior of $A = \eta(D+\gamma B)$ and B follows directly from the a priori estimates given above. The asymptotic order of C follows from (4.9) and $E_t = O(\overline{E}_t) = O(T-t)$ in L^∞ as $t \to T$. □

4.4. Existence

In this section we prove Theorem 4.2.3, i.e. the existence of a solution to the BSDE syetem (4.6) that satisfies the singular terminal condition (4.8). Similarly as in Chapter 3, our proof of existence is based on the asymptotic behavior established in Corollary 4.3.3. It suggests the following asymptotic ansatz:

$$\begin{aligned} A_t &= \frac{\eta}{T-t} + \frac{H_t}{(T-t)^2}, & H_t &= O((T-t)^2) \text{ in } L^\infty \text{ as } t \to T, \\ B_t &= 1 + \frac{G_t}{T-t}, & G_t &= O((T-t)^2) \text{ in } L^\infty \text{ as } t \to T, \\ C_t &= P_t, & P_t &= O((T-t)^2) \text{ in } L^\infty \text{ as } t \to T, \end{aligned} \tag{4.12}$$

where the asymptotic order of H and G is raised artificially for similar reasons as in Chapter 3, see Remark 3.3.2, to obtain the locally Lipschitz type statement given in Lemma 4.4.1(ii) below, while the reduced order of P unifies the notation and allows us to solve for all three processes in the same weighted L^∞-space.

The asymptotic ansatz (4.12) reduces the original system (4.6) to

$$\begin{cases} -dH_t = \left\{(T-t)^2\lambda_t - \dfrac{1}{\eta}\left(\dfrac{H_t}{T-t} - \gamma(T-t+G_t)\right)^2 + 2\gamma(T-t+G_t)\right\}dt \\ \qquad\quad - Z_t^H dW_t \\ -dG_t = \left\{-\rho_t(T-t+G_t) + \dfrac{1}{\eta}\left(\gamma P_t - \dfrac{G_t}{T-t}\right)\left(\dfrac{H_t}{T-t} - \gamma(T-t+G_t)\right) + \gamma P_t\right\}dt \\ \qquad\quad - Z_t^G dW_t \\ -dP_t = \left\{-2\rho_t P_t - \dfrac{1}{\eta}\left(\gamma P_t - \dfrac{G_t}{T-t}\right)^2\right\}dt - Z_t^P dW_t. \end{cases} \tag{4.13}$$

We define $f : \Omega \times [0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ such that, by identifying $Y = (H,G,P)$ and $Z = (Z^H, Z^G, Z^P)$, we have

$$-dY_t = f(t,Y_t)\,dt - Z_t\,dW_t$$

as a compact notation for (4.13). For $\delta > 0$ specified below, we will establish the existence of a short-time solution to (4.13) in the space

$$\mathcal{H} = \{Y \in L^\infty_{\mathcal{F}}(\Omega; C([T-\delta,T];\mathbb{R}^3)) : \|Y\|_{\mathcal{H}} < +\infty\}$$

endowed with the norm

$$\|Y\|_{\mathcal{H}} = \left\|(T-\cdot)^{-2}Y_\cdot\right\|_{L^\infty_{\mathcal{F}}(\Omega;C([T-\delta,T];\mathbb{R}^3))}.$$

Since $Y_T = 0$ this means that we are looking for a fixed point in $\mathcal{H}$ of the operator

$$\Gamma(Y) := \left(\mathbb{E}\left[\int_t^T f(s,Y_s)\,ds\,\middle|\,\mathcal{F}_t\right]\right)_{T-\delta\le t\le T}.$$

Lemma 4.4.1. *The following holds:*

(i) $\mathcal{H}$ is complete.

(ii) For every $R > 0$ there exists a constant $L > 0$ (independent of δ) such that

$$\|f(\cdot,Y_\cdot) - f(\cdot,X_\cdot)\|_{\mathcal{H}} \le L\|Y_\cdot - X_\cdot\|_{\mathcal{H}} \quad \text{for all } Y,X \in \overline{B}_{\mathcal{H}}(R).$$

Proof. The spaces $L^\infty_{\mathcal{F}}(\Omega; C([T-\delta,T];\mathbb{R}))$ and $\mathcal{H}$ are isometrically isomorphic by identifying $Y \in L^\infty_{\mathcal{F}}(\Omega; C([T-\delta,T];\mathbb{R}))$ with the process $((T-t)^2 Y_t)_{T-\delta\le t\le T}$ in $\mathcal{H}$. Hence $\mathcal{H}$ is complete.

In order to establish the Lipschitz continuity, let $\overline{Y_tY_t'}$ be the line segment connecting Y_t and Y_t'. By the mean value theorem we have for $Y, Y' \in \overline{B}_{\mathcal{H}^2}(R)$, $d\mathbb{P} \times dt$-a.e.,

$$\begin{aligned}|f(t,Y_t) - f(t,Y_t')| &\le \sup_{y \in \overline{Y_tY_t'}} \|\partial_y f(t,y)\|_{\mathrm{Hom}(\mathbb{R}^3;\mathbb{R}^3)} |Y_t - Y_t'| \\ &\le (T-t)^2 \sup_{|y| \le (T-t)^2 R} \|\partial_y f(t,y)\|_{\mathrm{Hom}(\mathbb{R}^3;\mathbb{R}^3)} \|Y - Y\|_{\mathcal{H}}, \quad (4.14)\end{aligned}$$

where it is used that the line $\overline{Y_tY_t'}$ is contained in $\overline{B}_{\mathbb{R}^3}((T-t)^2R)$, $d\mathbb{P} \times dt$-a.e. But,

$$\partial_y f(t,y) =$$
$$\begin{pmatrix} \frac{-2y_1}{\eta(T-t)^2} + \frac{2\gamma y_2}{\eta(T-t)} + \frac{2\gamma}{\eta} & \frac{2\gamma y_1}{\eta(T-t)} - \frac{2(y_2+T-t-\eta\gamma^{-1})}{\eta\gamma^{-2}} & 0 \\ \frac{-y_2}{\eta(T-t)^2} + \frac{\gamma y_3}{\eta(T-t)} & \frac{-y_1}{\eta(T-t)^2} + \frac{2\gamma y_2}{\eta(T-t)} - \frac{\gamma y_3 - 1}{\eta\gamma^{-1}} - \rho_t & \frac{\gamma y_1}{\eta(T-t)} - \frac{y_2+T-t-1}{\eta\gamma^{-2}} \\ 0 & \frac{-2y_2}{\eta(T-t)^2} + \frac{2\gamma y_3}{\eta(T-t)} & \frac{2\gamma y_2}{\eta(T-t)} - \frac{2y_3}{\eta\gamma^{-2}} - 2\rho_t \end{pmatrix},$$

from which we see that the supremum in (4.14) is essentially bounded on $\Omega \times [T-\delta, T]$. □

Choosing R and δ appropriately the preceding lemma allows us to use a standard fix-point argument to show that Γ has a unique fix-point. The fix-point is just a local solution to (4.13).

Proposition 4.4.2. *For $\delta > 0$ sufficient small there exists a short-time solution $(Y,Z) \in \mathcal{H}^2 \times L^2_{\mathcal{F}}(T-\delta, T; \mathbb{R}^{3\times m})$ to (4.13).*

Proof. Let us fix $R = 4\max\{T\|\lambda\|_{L^\infty} + \gamma^2 T/\eta + 2\gamma, \|\rho\|_{L^\infty}\}$ and choose $L > 0$ as in Lemma 4.4.1. For $Y, Y' \in \overline{B}_{\mathcal{H}}(R)$ it then holds $d\mathbb{P} \times dt$-a.e.,

$$\begin{aligned}|\Gamma(Y)_t - \Gamma(Y')_t| &\le \mathbb{E}\left[\int_t^T |f(s,Y_s) - f(s,Y_s')|\, ds \middle| \mathcal{F}_t\right] \\ &\le L(T-t)^3 \|Y - Y'\|_{\mathcal{H}}\end{aligned}$$

This yields, as long as $0 < \delta \le (2L)^{-1}$,

$$\|\Gamma(Y) - \Gamma(Y')\|_{\mathcal{H}} \le \frac{1}{2}\|Y - Y'\|_{\mathcal{H}}.$$

Hence, Γ is an 1/2-contraction on $\overline{B}_{\mathcal{H}}(R)$. Furthermore, Γ maps $\overline{B}_{\mathcal{H}}(R)$ onto itself. Indeed, for all $Y \in \overline{B}_{\mathcal{H}}(R)$ it holds $d\mathbb{P} \times dt$-a.e.,

$$\begin{aligned}|\Gamma(Y)_t| &\le |\Gamma(Y)_t - \Gamma(0)_t| + |\Gamma(0)_t| \\ &\le (T-t)^2 \frac{R}{2} + \mathbb{E}\left[\int_t^T |f(s,0)|\, ds \middle| \mathcal{F}_t\right] \\ &\le (T-t)^2\frac{R}{2} + 2(T-t)^2 \max\{T\|\lambda\|_{L^\infty} + \frac{\gamma^2}{\eta}T + 2\gamma, \|\rho\|_{L^\infty}\} = (T-t)^2 R.\end{aligned}$$

As a result, Γ has a unique fixed point $Y \in \overline{B}_{\mathcal{H}}(R)$. The process Y satisfies

$$Y_t = -\int_{T-\delta}^{t} f(s, Y_s)\, ds + \mathbb{E}\left[\int_{T-\delta}^{T} f(s, Y_s)\, ds \,\middle|\, \mathcal{F}_t \right].$$

By the martingale representation theorem there exits $Z \in L^2_{\mathcal{F}}(T-\delta, T; \mathbb{R}^{3\times m})$ such that

$$Y_t = -\int_{T-\delta}^{t} f(s, Y_s)\, ds + \int_{T-\delta}^{t} Z_s\, dW_s.$$

Hence, (Y, Z) gives the desired short-time solution to (4.13). $\square$

We are now ready to prove Theorem 4.2.3.

Proof of Theorem 4.2.3. The short-time solution to (4.13) established by Proposition 4.4.2 gives in terms of the ansatz (4.12) a short-time solution

$$(A, B, C) \in L^\infty_{\mathcal{F}}(\Omega; ([T-\delta, T^-]; \mathbb{R}^3)) \times L^2_{\mathcal{F}}(T-\delta, T^-; \mathbb{R}^{3\times m})$$

to (4.6) that satisfies the singular terminal condition (4.8). In order to see that this short-time solution extends to a global solution in $L^\infty_{\mathcal{F}}(\Omega; C([0, T^-]; \mathbb{R}^3)) \times L^2_{\mathcal{F}}(0, T^-; \mathbb{R}^{3\times m})$ notice first that the system (4.6) satisfies the assumptions of the local L^∞-existence results for BSDEs with locally Lipschitz drivers of Lemma A.3.2 given in the appendix. Hence, the system (4.6) imposed with the essentially bounded terminal value $(A_{T-\delta}, B_{T-\delta}, C_{T-\delta})$ admits an essentially bounded local extension on $[T-\delta-\delta', T-\delta]$.

Due to the a priori estimates given in Proposition 4.3.2 we know that this local extension will stay (recalling $A = \eta(D + \gamma B)$) in the bounded region $[0, \eta(\overline{D}_{T-\delta} + \gamma)] \times [0, 1] \times [-1/\gamma, 0]$. When iterating this extension procedure we may therefore choose (cf. the proof of Lemma A.3.2) step by step the same local Lipschitz constant $L > 0$ for the system (4.6), which results in a constant length $\delta' > 0$ of the extension interval. Thus, after finitely many steps we obtain a global extension on $[0, T)$. $\square$

4.5. Verification

This section devoted to the verification statement of Theorem 4.2.4. Throughout, let

$$((A, B, C), (Z^A, Z^B, Z^C)) \in L^\infty_{\mathcal{F}}(0, T^-; \mathbb{R}^3) \times L^2_{\mathcal{F}}(0, T^-; \mathbb{R}^{3\times m})$$

denote any solution to (4.6) that satisfies (4.8) and recall that the candidate optimal strategy ξ^* is given in terms of the processes

$$D := \eta^{-1}(A - \gamma B) \qquad \text{and} \qquad E := \eta^{-1}(\gamma C - B + 1)$$

for which a priori estimates have been established in Section 4.3. The proof of the admissibility of ξ^* uses the following iterated integral version of Gronwall's inequality.

Lemma 4.5.1 ([BS92, Corollary 11.1])**.** *Let $u(t)$, $a(t)$, and $b(t)$ be nonnegative continuous functions on $[0,T]$ with $a(t)$ and $b(t)$ being nondecreasing, and suppose*

$$u(t) \leq a(t) + b(t) \int_0^t \int_0^s k(s,r)u(r)\,dr\,ds, \qquad 0 \leq t \leq T,$$

where $k(s,r)$ is a nonnegative continuous function on $\{0 \leq r \leq s \leq T\}$. Then

$$u(t) \leq a(t) \exp\left(b(t) \int_0^t \int_0^s k(s,r)\,dr\,ds\right), \qquad 0 \leq t \leq T.$$

We are now ready to verify that the candidate optimal control ξ^* is indeed admissible.

Lemma 4.5.2. *The feedback control ξ^* given in (4.7) is admissible.*

Proof. Let us fix an initial state $(t,x,y) \in [0,T) \times \mathbb{R} \times \mathbb{R}$. The dynamics of the state process (X^*, Y^*) under the candidate ξ^* is given by:

$$\begin{cases} dX_s^* = \{-D_s X_s^* + E_s Y_s^*\}\,ds \\ dY_s^* = \{-(\rho_s + \gamma E_s) Y_s^* + \gamma D_s X_s^*\}\,ds. \end{cases} \tag{4.15}$$

Due to the singularity of D at the terminal time, it is not clear yet that the solution to (4.15) is well-defined at the terminal time; a priori we only know that $(X^*, Y^*) \in L^\infty_{\mathcal{F}}(\Omega; C([t,T^-];\mathbb{R}^2)$.

In order to show that $X_T^* = 0$ we first apply the variation of constants formula for $t \leq s < T$ to get:

$$X_s^* = xe^{-\int_t^s D_u\,du} + \int_t^s e^{-\int_r^s D_u\,du} E_r Y_r^*\,dr$$

and

$$Y_s^* = ye^{-\int_t^s (\rho_u + \gamma E_u)\,du} + \int_t^s e^{-\int_r^s (\rho_u + \gamma E_u)\,du} \gamma D_r X_r^*\,dr. \tag{4.16}$$

Hence, the process $\tilde{X}_s := X_s^* e^{\int_t^s D_r\,dr}$ satisfies,

$$\tilde{X}_s = x + \int_t^s e^{\int_t^r D_u\,du} E_r Y_r^*\,dr.$$

Since $\rho, E \geq 0$, this yields,

$$\begin{aligned} |\tilde{X}_s| &\leq |x| + \int_t^s e^{\int_t^r D_u\,du} E_r |y|\,dr + \int_t^s e^{\int_t^r D_u\,du} E_r \int_t^r \gamma D_u e^{-\int_t^u D_v\,dv} |\tilde{X}_u|\,du\,dr \\ &= |x| + |y| \int_t^s e^{\int_t^r D_u\,du} E_r\,dr + \int_t^s \gamma E_r \int_t^r D_u e^{\int_u^r D_v\,dv} |\tilde{X}_u|\,du\,dr. \end{aligned}$$

By the iterated integral version of Gronwall's inequality (Lemma 4.5.1),

$$\begin{aligned} |\tilde{X}_s| &\leq \left(|x| + |y| \int_t^s e^{\int_t^r D_u\,du} E_r\,dr\right) \exp\left(\int_t^s \gamma E_r \int_t^r D_u e^{\int_u^r D_v\,dv}\,du\,dr\right) \\ &= \left(|x| + |y| \int_t^s e^{\int_t^r D_u\,du} E_r\,dr\right) \exp\left(\int_t^s \gamma E_r \left(e^{\int_t^r D_u\,du} - 1\right) dr\right). \end{aligned} \tag{4.17}$$

In view of the a priori upper bounds on D and E, because the antiderivative of $\coth(\cdot)$ is given by $\ln(\sinh(\cdot))$ and because $\cosh(\cdot) \geq 1$,

$$\begin{aligned}\int_t^s \gamma E_r e^{\int_t^r D_u\,du}\,dr &\leq \int_t^s \kappa\tanh(\kappa(T-r))e^{\int_t^r \kappa\coth(\kappa(T-u))\,du}\,dr\\ &= \int_t^s \kappa\tanh(\kappa(T-r))\frac{\sinh(\kappa(T-t))}{\sinh(\kappa(T-r))}\,dr\\ &\leq \kappa(s-t)\sinh(\kappa(T-t))\\ &\leq \kappa T\sinh(\kappa T).\end{aligned}$$

Along with (4.17) this shows that $|\tilde{X}_s|$ is bounded as $s \to T$. Therefore, this time using the a priori lower bound for D,

$$\begin{aligned}|X_s^*| &= |\tilde{X}_s|\exp\left(-\int_t^s D_r\,dr\right)\\ &\leq |\tilde{X}_s|\exp\left(-\int_t^s \frac{\eta^{-1}\gamma}{e^{\eta^{-1}\gamma(T-r)}-1}\,dr\right)\\ &= |\tilde{X}_s|\frac{1-e^{-\eta^{-1}\gamma(T-s)}}{1-e^{-\eta^{-1}\gamma(T-t)}} \xrightarrow{s\to T} 0.\end{aligned} \tag{4.18}$$

This shows that $X_T^* = 0$. It also shows that $X_s^* = O(T-s)$ in L^∞ as $s \to T$. As $D_s = O((T-s)^{-1})$ it follows that

$$DX^* \in L^\infty_{\mathcal{F}}(\Omega; C([t,T];\mathbb{R})).$$

The boundedness of DX^* again implies by (4.16) that $Y^* \in L^\infty_{\mathcal{F}}(\Omega; C([t,T];\mathbb{R}))$. Hence, we conclude

$$\xi^* \in L^\infty_{\mathcal{F}}(\Omega; C([t,T];\mathbb{R})).$$

This proves that ξ^* is indeed admissible. □

Lemma 4.5.3. *For every $\xi \in \mathcal{A}(t,x)$ it holds $E[A_sX_s^2+B_sX_sY_s+C_sY_s^2|\mathcal{F}_t] \xrightarrow{s\to T} 0$.*

Proof. Recalling $B, C \in L^\infty_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}))$, $X, Y \in L^2_{\mathcal{F}}(\Omega; C([t,T];\mathbb{R}))$, and $X_T = C_T = 0$, it follows by the dominated convergence theorem,

$$\mathbb{E}[B_sX_sY_s + C_sY_s^2|\mathcal{F}_t] \xrightarrow{s\to T} 0.$$

Furthermore, note that by $X_T = 0$ and Jensen's inequality,

$$X_s^2 = \left(\int_s^T \xi_r\,dr\right)^2 \leq (T-s)\int_s^T \xi_r^2\,dr.$$

Hence, by Corollary 4.3.3,

$$\mathbb{E}[A_sX_s^2|\mathcal{F}_t] \leq \mathbb{E}\left[(T-s)A_s\int_s^T \xi_r^2\,dr\,\middle|\,\mathcal{F}_t\right] \xrightarrow{s\to T} 0.$$ □

We are now ready to prove the verification theorem.

Proof of Theorem 4.2.4. By a slight abuse of notation we *define* within this proof the random fields $V_t(x,y)$ and $Z_t(x,y)$ by the linear-quadratic ansatz (4.5) and verify that this gives indeed the value function of the control problem. For the moment we only know that (V,Z) solves the HJB equation (4.4).

Let us fix an initial state $(t,x,y) \in [0,T) \times \mathbb{R} \times \mathbb{R}$ and admissible control $\xi \in \mathcal{A}(t,x)$. For $n \in \mathbb{N}$ we define the stopping time

$$\tau_n := \inf\{t \leq s \leq T : |X_s| \vee |Y_s| \geq n\}.$$

Since (V,Z) solve the HJB equation, it holds by the Itô-Kunita formula [Kun84, Theorem I.8.1] for all $t \leq s < T$,

$$\begin{aligned} V_t(x,y) = {} & V_{s\wedge\tau_n}(X_{s\wedge\tau_n}, Y_{s\wedge\tau_n}) \\ & + \int_t^{s\wedge\tau_n} \{\xi_r \partial_x V_r(X_r,Y_r) - (-\rho_r Y_r + \gamma\xi_r)\partial_y V_r(X_r,Y_r)\}\, dr \\ & + \int_t^{s\wedge\tau_n} \inf_{\xi\in\mathbb{R}} \{-\xi\partial_x V_r(X_r,Y_r) - (\rho_t Y_r - \gamma\xi)\partial_y V_r(X_r,Y_r) \\ & \qquad\qquad + \tfrac{1}{2}\eta\xi^2 + \xi y + \tfrac{1}{2}\lambda_t x^2\}\, dr \\ & - \int_t^{s\wedge\tau_n} Z_r(X_r,Y_r)\, dW_r. \end{aligned} \tag{4.19}$$

The above stochastic integral stopped at τ_n is a true martingale. Hence,

$$V_t(x,y) \leq \mathbb{E}[V_{s\wedge\tau_n}(X_{s\wedge\tau_n}, Y_{s\wedge\tau_n})|\mathcal{F}_t] + \mathbb{E}\left[\int_t^{s\wedge\tau_n} \{\tfrac{1}{2}\eta\xi_r^2 + \xi_r Y_r + \tfrac{1}{2}\lambda_r X_r^2\}\, dr \,\middle|\, \mathcal{F}_t\right]. \tag{4.20}$$

Since the coefficients A, B, C of the random field $V_r(x,y)$ are essentially bounded on $[t,s]$, since $X, Y \in L^2_{\mathcal{F}}(\Omega; C([t,T];\mathbb{R}))$, and because $\xi \in L^2_{\mathcal{F}}(t,T;\mathbb{R})$ and $\lambda \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}_+)$, it follows by Hölder's inequality that

$$AX^2, BXY, CY^2 \in L^1_{\mathcal{F}}(\Omega; C([t,s];\mathbb{R})) \quad \text{and} \quad \xi^2, \xi Y, \lambda X^2 \in L^1_{\mathcal{F}}(t,T;\mathbb{R}).$$

Hence, the dominated convergence theorem applies when letting $n \to \infty$ in (4.20), which yields,

$$V_t(x,y) \leq \mathbb{E}[V_s(X_s,Y_s)|\mathcal{F}_t] + \mathbb{E}\left[\int_t^s \{\tfrac{1}{2}\eta\xi_r^2 + \xi_r Y_r + \tfrac{1}{2}\lambda_r X_r^2\}\, dr \,\middle|\, \mathcal{F}_t\right]. \tag{4.21}$$

Hence, by Lemma 4.5.3 and again the dominated convergence theorem letting $s \to T$ yields,

$$V_t(x,y) \leq \mathbb{E}\left[\int_t^T \tfrac{1}{2}\eta\xi_r^2 + \xi_r Y_r + \tfrac{1}{2}\lambda_r X_r^2\, dr \,\middle|\, \mathcal{F}_t\right]. \tag{4.22}$$

Finally note that since the feedback control ξ^* attains the infimum in (4.19) it holds equality in (4.20)–(4.22) if $\xi = \xi^*$. $\square$

4.6. Conclusion

In this chapter we analyzed a novel stochastic optimal control problem arising in models of optimal trade execution with instantaneous and persistent price impact and stochastic resilience. Assuming that the instantaneous impact factor is constant but allowing for stochastic resilience and market risk we characterized the value function in terms of the unique solution to a three-dimensional stochastic Riccati equation with singular terminal condition in the first component. Our existence of solutions results used an extension of the asymptotic expansion approach introduced in [GHS16] to a multi-dimensional setting. Several open problems remain. First, we cannot guarantee non-negativity of the trading rate. Intuitively, price-triggered round trips should not be beneficial. Based on our analysis, they can not be ruled out, though. Second, the assumption of a constant instantaneous impact factor was important to establish the a priori estimates. An extension to more general impact factors is certainly desirable. Third, a numerical analysis of a deterministic benchmark model suggests that our model can be viewed as a approximation to a model with both absolutely continuous and singular controls if $\eta \to 0$. While a formal proof of this limit result in a general non-Markovian framework would certainly be desirable it is clearly beyond the scope of the present paper.

A. Appendix

A.1. Three results on BSPDEs

For the reader's convenience this appendix recalls three results on BSPDEs which are used throughout this paper. The following existence and uniqueness of solutions result for BSPDEs is established in [DQT12, Theorem 5.5].

Proposition A.1.1 ([DQT12, Theorem 5.5]). *Let the coefficients b, σ and $\bar{\sigma}$ satisfy the assumptions $(\mathcal{A}1) - (\mathcal{A}3)$. Suppose that the random function $f(\cdot,\cdot,\cdot,\vartheta,y,z) \in \mathcal{L}^2_{\mathcal{F}}(0,T;L^2(\mathbb{R}^d))$ for any $(\vartheta,y,z) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^m$ and that there exists a positive constant L_0 such that for all $(\vartheta_1,y_1,z_1),(\vartheta_2,y_2,z_2) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^m$ and $(\omega,t,x) \in \Omega\times[0,T]\times\mathbb{R}^d$,*

$$|f(\omega,t,x,\vartheta_1,y_1,z_1) - f(\omega,t,x,\vartheta_2,y_2,z_2)| \le L_0(|\vartheta_1-\vartheta_2| + |y_1-y_2| + |z_1-z_2|).$$

Then, for any given $G \in L^2(\Omega,\mathcal{F}_T;H^{k,2}(\mathbb{R}^d))$ with $k \in \{0,1\}$, the BSPDE

$$\begin{cases} -du_t(x) = \ \{\mathrm{tr}\left(a_t(x)D^2u_t(x) + D\psi_t(x)\sigma_t^{\mathcal{T}}(x)\right) + b_t^{\mathcal{T}}(x)Du_t(x) \\ \qquad\qquad + f(t,x,x_t(x),Du_t(x),\psi_t(x))\}\,dt - \psi_t(x)\,dW_t, \quad (t,x)\in[0,T]\times\mathbb{R}^d; \\ u_T(x) = G(x), \quad x\in\mathbb{R}^d, \end{cases} \tag{A.1}$$

admits a unique solution $(u,\psi) \in \mathcal{H}^k \times \mathcal{L}^2_{\mathcal{F}}(0,T;H^{k,2}(\mathbb{R}^d))$, i.e., it holds almost surely that

$$\begin{aligned} \langle\varphi,\,u_t\rangle = \langle\varphi,\,u_T\rangle + &\int_t^T \left\{\langle\varphi,\,\mathrm{tr}\left(a_sD^2u_s + D\psi_s\sigma_s^{\mathcal{T}}\right) + b_s^{\mathcal{T}}Du_s + f(s,u_s,Du_s,\psi_s)\rangle\right\}ds \\ &- \int_t^T \langle\varphi,\,\psi_s dW_s\rangle \quad \forall\varphi\in C_c^\infty(\mathbb{R}^d), \quad t\in[0,T], \end{aligned} \tag{A.2}$$

where $C_c^\infty(\mathbb{R}^d)$ is the set of all the infinitely differentiable functions with compact supports on $\mathbb{R}^d$. Moreover, $u \in \mathcal{L}^{2,\infty}_{\mathcal{F}}(0,T)$ if $k=1$, and there exists a constant C that depends only on κ, L, L_0, Λ and T such that

$$\begin{aligned} &\|u\|_{\mathcal{H}^k} + \|u\|_{\mathcal{L}^{2,\infty}_{\mathcal{F}}(0,T)}1_{k=1} + \|\psi\|_{\mathcal{L}^2_{\mathcal{F}}(0,T;H^{k,2}(\mathbb{R}^d))} \\ &\qquad \le C\left(\|f(\cdot,\cdot,\cdot,0,0,0)\|_{\mathcal{L}^2_{\mathcal{F}}(0,T;L^2(\mathbb{R}^d))} + \|G\|_{L^2(\Omega,\mathcal{F}_T;H^{k,2}(\mathbb{R}^d))}\right). \end{aligned} \tag{A.3}$$

By using the standard denseness arguments, it is easily checked that for $k=1$, the requirement by (A.2) with test functions for the definition of solution is equivalent to the corresponding one holding almost everywhere in Definition 2.2.1. The nonlinear term f in Proposition A.1.1 can be rewritten in linear form as

$$f(t,x,\vartheta,y,z) = \alpha\vartheta + \beta^{\mathcal{T}} y + \vartheta^{\mathcal{T}} z + f(t,x,0,0,0), \quad (\vartheta,y,z) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^m, \tag{A.4}$$

where

$$\begin{aligned}
\alpha &= \frac{f(t,x,\vartheta,y,z) - f(t,x,0,y,z)}{\vartheta} 1_{\vartheta\neq 0};\\
\beta^i &= \frac{f(t,x,0,y^{(i)},z) - f(t,x,0,y^{(i-1)},z)}{y_i} 1_{y_i\neq 0}, \quad i=1,\dots,d;\\
\vartheta^k &= \frac{f(t,x,0,0,z^{(k)}) - f(t,x,0,0,z^{(k-1)})}{z_k} 1_{z_k\neq 0}, \quad k=1,\dots,m;\\
y^{(i)} &= (y_1,\dots,y_i,0,\dots,0), \quad y^{(0)} = 0\in\mathbb{R}^d, \quad i=1,\dots,d;\\
z^{(k)} &= (z_1,\dots,z_k,0,\dots,0), \quad z^{(0)} = 0\in\mathbb{R}^m, \quad k=1,\dots,m.
\end{aligned}$$

Thus, the comparison principle for linear BSPDEs [DQT12, Theorem 6.3] implies immediately the following result.

Corollary A.1.2 (Corollary of [DQT12, Theorem 6.3])**.** *Under the hypothesis of Proposition A.1.1, for $k=1$, suppose the pair (G',f') satisfies the same conditions as (G,f) in Proposition A.1.1. Let (u,v) and (u',v') be the respective solutions to the BSPDE* (A.1) *and assume furthermore that for almost every $(\omega,t,x)\in\Omega\times[0,T]\times\mathbb{R}^d$ it holds*

$$f(\omega,t,x,u_t,Du_t,v) \geq f'(\omega,t,x,u_t,Du_t,v) \quad \text{and} \quad G(\omega,x)\geq G'(\omega,x).$$

Then, $u\geq u'$, $\mathbb{P}\otimes dt\otimes dx$-a.e.

The corollary can be verified by applying the linearization (A.4) to the function

$$\tilde{f}(t,x,\vartheta,y,z) := f'(\omega,t,x,u_t,Du_t,v) - f'(t,x,u_t+\vartheta,Du_t+y,v+z).$$

The proof is standard and hence omitted. We close this appendix with the following lemma on an inequality for the positive part of the solutions to BSPDEs, whose proof will be sketched below.

Lemma A.1.3. *Let $u\in\mathcal{H}^0$. Suppose that for any $\varphi\in C_c^\infty(\mathbb{R}^d)$, almost surely*

$$\langle\varphi,\, u_t\rangle = \langle\varphi,\, G\rangle + \int_t^T \left\{ \langle\varphi,\, h_s+f_s\rangle - \sum_{i=1}^d \langle\partial_{x^i}\varphi,\, g_s^i\rangle \right\} ds - \int_t^T \langle\varphi,\, \zeta_s\, dW_s\rangle,$$

where $G\in L^2(\Omega,\mathcal{F}_T,L^2(\mathbb{R}^d))$; $\zeta,f,g\in\mathcal{L}^2_{\mathcal{F}}(0,T;L^2(\mathbb{R}^d))$ and $h\in\mathcal{L}^1_{\mathcal{F}}(0,T;L^1(\mathbb{R}^d))$. Moreover, assume $h_s(x)u_s^+(x)\leq 0$, $\mathbb{P}\otimes dt\otimes dx$-a.e. Then, it holds almost surely *that*

$$\begin{aligned}&\|u_t^+\|^2_{L^2(\mathbb{R}^d)} + \int_t^T \|\zeta_s 1_{u>0}\|^2_{L^2(\mathbb{R}^d)}\, ds \\ &\quad\le \|G^+\|^2_{L^2(\mathbb{R}^d)} + 2\int_t^T \left\{\langle u_s^+,\, f_s\rangle - \sum_{i=1}^d \langle \partial_{x^i} u_s^+,\, g_s^i\rangle\right\} ds - 2\int_t^T \langle u_s^+,\, \zeta_s\, dW_s\rangle.\end{aligned} \tag{A.5}$$

Sketch of the proof. The pair (u, ζ) is the unique solution in $\mathcal{H}^0 \times \mathcal{L}^2_{\mathcal{F}}(0,T; L^2(\mathbb{R}^d))$ to the linear BSPDE

$$\begin{cases} -du_t(x) = \left\{\Delta u_t(x) + f_t + h_t + \displaystyle\sum_{i=1}^d \partial_{x^i}(g_t^i - \partial_{x^i} u_t(x))\right\} dt \\ \qquad\qquad - \zeta_t(x)\, dW_t, \quad (t,x) \in [0,T] \times \mathbb{R}^d; \\ u_T(x) = G(x), \quad x \in \mathbb{R}^d. \end{cases}$$

If $h \in \mathcal{L}^2_{\mathcal{F}}(0,T; L^2(\mathbb{R}^d))$, then (A.5) follows from [QW14, Corollary 3.11]. For $h \in \mathcal{L}^1_{\mathcal{F}}(0,T; L^1(\mathbb{R}^d))$, it can be verified using a standard approximation method. To this end, we first observe that the proof of [DMS09, Proposition 2] of the Itô formula for *forward* SPDEs is independent of the boundedness of the domain $\mathcal{O}$ therein and hence the result extends to $\mathcal{O} = \mathbb{R}^d$. Thus, for any function $\Phi : \mathbb{R} \to \mathbb{R}$ with bounded derivatives Φ' and Φ'' and $\Phi'(0) = 0$, it holds almost surely that

$$\begin{aligned}&\int_{\mathbb{R}^d} \Phi(u_t(x))\, dx + \frac{1}{2}\sum_{r=1}^m \int_t^T \langle \Phi''(u_s)\zeta_s^r,\, \zeta_s^r\rangle\, ds + \int_t^T \langle \Phi'(u_s),\, \zeta_s\, dW_s\rangle \\ &\quad = \int_{\mathbb{R}^d} \Phi(G(x))\, dx + \int_t^T \left\{\langle \Phi'(u_s),\, f_s + h_s\rangle - \sum_{i=1}^d \langle \Phi''(u_s)\partial_{x^i} u_s,\, g_s^i\rangle\right\} ds.\end{aligned} \tag{A.6}$$

If $\Phi'(y) = \Phi'(y)1_{(0,\infty)}(y) \ge 0$, then our assumptions on h yield almost surely that

$$\text{LHS of (A.6)} \le \int_{\mathbb{R}^d} \Phi(G(x))\, dx + \int_t^T \left\{\langle \Phi'(u_s),\, f_s\rangle\, ds - \sum_{i=1}^d \langle \Phi''(u_s)\partial_{x^i} u_s,\, g_s^i\rangle\right\} ds. \tag{A.7}$$

We can generalize the above inequality to Φ' being unbounded, by approximating Φ and passing to the limit in (A.7). Then it remains to apply inequality (A.7) to the function $\Psi : y \mapsto (y^+)^2$. Though Ψ is not regular enough, this can be done using the same approximation method as in Step 2 of the proof of [QT12, Lemma 3.5]. □

A.2. Comparison principle for viscosity solutions

In this appendix we present a modification of the comparison result for viscosity solutions given in [BBP97]. The original statement [BBP97, Theorem 3.5] concerns the uniqueness of viscosity solutions to systems of semilinear parabolic equations. The related comparison result is mentioned in [BBP97, Remark 3.9]. As suggested in [PR14, Remark 6.105], the present scalar formulation covers the case of a monotone (not necessarily Lipschitz continuous) nonlinearity $G : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$. Namely, we assume

(A5) G is continuous,

(A6) $(u-v)(G(t,y,u) - G(t,y,v)) \leq \mu(u-v)^2$ for $\mu \in \mathbb{R}$ uniform in $t \in [0,T]$, $y \in \mathbb{R}^d$, $u, v \in \mathbb{R}$,

and consider for given terminal value $g : \mathbb{R}^d \to \mathbb{R}$ the following parabolic problem

$$\begin{cases} -\partial_t u(t,y) - \mathcal{L}u(t,y) - G(t,y,u(t,y)) = 0, & (t,y) \in [0,T) \times \mathbb{R}^d, \\ u(T,y) = g(y), & y \in \mathbb{R}^d. \end{cases} \tag{A.8}$$

It is worth mentioning that no uniform continuity type assumptions are needed due to the lack of any gradient dependence of G.

Theorem A.2.1. *Let (A5) and (A6) hold, and let $\underline{u}, \overline{u} \in C_{poly}([0,T] \times \mathbb{R}^d)$ be a viscosity sub- and a viscosity supersolution to* (A.8)*, respectively. Then, $\underline{u} \leq \overline{u}$ in $[0,T] \times \mathbb{R}^d$.*

The proof is as in [BBP97]. The only modification needed is to linearize the difference $G(t,y,\underline{u}(t,y)) - G(t,y,\overline{u}(t,y))$ in terms of

$$l(t,y) := 1_{\underline{u}(t,y) \neq \overline{u}(t,y)} \frac{G(t,y,\underline{u}(t,y)) - G(t,y,\overline{u}(t,y))}{\underline{u}(t,y) - \overline{u}(t,y)}.$$

rather than estimating it by a Lipschitz property, cf. [BBP97, p.78]. The key lemma [BBP97, Lemma 3.7] based on the theorem of sums and the doubling variable technique then reads:

Lemma A.2.2. *The difference $w := \underline{u} - \overline{u}$ is a viscosity subsolution to the linear equation*

$$-\partial_t w(t,y) - \mathcal{L}w(t,y) - l(t,y)w(t,y) = 0, \qquad (t,y) \in [0,T) \times \mathbb{R}^d.$$

Since l is by (A6) bounded above by μ the supersolution property of the function given in [BBP97, Lemma 3.8] carries over to equation (A.2.2) so that the rest of the proof matches again with the original reference.

A.3. Comparison principle for multi-dimensional BSDEs

A necessary and sufficient condition under which the comparison theorem holds for multi-dimensional BSDEs has been first given by Hu and Peng [HP06]. The equivalent quasi-monotonicity condition (iv) below can be found in [Xu16, Theorem 3.1]. The comparison results in [HP06, Xu16] are stated under an additional continuity condition on the drivers that is not satisfied in our model. However, the continuity condition is only needed to prove that if a comparison principle holds, then the system is necessarily quasi-monotone. Continuity is not needed for the converse implication. As such, their results are in fact applicable to our framework. Even though, for the reader's convenience we refer instead to a comparison result for multi-dimensional reflected BSDEs by Wu and Xiao [WX10] that is formulated explicitly under the weaker regularity assumption (i) given below.

Proposition A.3.1 ([WX10, Theorem 3.1]). *Let $(Y^i, Z^i) \in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^d)) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{d\times m})$, $i = 1, 2$, be solutions to the BSDEs*

$$-dY^i_t = f^i(t, Y^i_t, Z^i_t)\, dt - Z^i_t\, dW_t, \qquad 0 \le t \le T,$$

with the drivers $f^i : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times m} \to \mathbb{R}^d$, $i = 1, 2$, satisfying

(i) $f^i(\cdot, y, z) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ for all $y \in \mathbb{R}$ and $z \in \mathbb{R}^{d\times m}$,

(ii) there exits $L > 0$ such that for all $y, y' \in \mathbb{R}^d$ and $z, z' \in \mathbb{R}^{d\times m}$,

$$|f^i(t,y,z) - f^i(t,y',z')| \le L(|y-y'| + |z-z'|), \qquad d\mathbb{P}\times dt\text{-a.e.},$$

and suppose, in addition,

(iii) $Y^1_T \le Y^2_T$,

(iv) for every $k = 1, \ldots, d$ it holds for all $y^1, y^2 \in \mathbb{R}^d$ and $z^1, z^2 \in \mathbb{R}^{d\times m}$ such that $y^1_k = y^2_k$, $z^1_k = z^2_k$, $y^1_l \le y^2_l$, $l \ne k$:

$$f^1_k(t, y^1, z^1) \le f^2_k(t, y^2, z^2), \qquad d\mathbb{P}\times dt\text{-a.e.}$$

Then $Y^1_t \le Y^2_t$, $t \in [0,T]$.

Below we state a local L^∞-existence result for BSDEs with locally Lipschitz drivers not depending on Z. The result seems well well-known; we give it for completeness. Specifically, we consider the BSDE

$$Y_t = \zeta + \int_t^T f(s, Y_s)\, ds - \int_t^T Z_s\, dW_s, \qquad 0 \le t \le T, \tag{A.9}$$

where we assume that the terminal value

- $\zeta \in L^\infty_{\mathcal{F}_T}(\mathbb{R}^d)$

is essentially bounded and that the driver $f : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

- $f(\cdot, 0) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^d)$,
- for every $R > 0$ there exists $L > 0$ such that for all $|y|, |y'| \leq R$,

$$|f(t,y) - f(t,y')| \leq L|y - y'|. \tag{A.10}$$

Lemma A.3.2. *Under the above assumptions there exits $\delta > 0$ such that there exits on $[T-\delta,T]$ a short-time solution $(Y,Z) \in L^\infty_{\mathcal{F}}(\Omega; C([T-\delta,T];\mathbb{R}^d)) \times L^2_{\mathcal{F}}(T-\delta,T;\mathbb{R}^{d\times m})$ to* (A.9).

Proof. We will show that one may choose $\delta = 1/(2L)$, where L is the Lipschitz constant given in (A.10) with respect to $R = 2(\|\eta\|_{L^\infty} + T\|f(\cdot,0)\|_{L^\infty})$.

On $\mathcal{H} = L^\infty_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^d))$ we define the operator $\Gamma : \mathcal{H} \to \mathcal{H}$ by

$$\Gamma(Y)_t = \mathbb{E}\left[\zeta + \int_t^T f(s,Y_s)\,ds \,\middle|\, \mathcal{F}_t\right].$$

Then Γ is a contraction on $\overline{B}_{\mathcal{H}}(R)$: For all $Y, Y' \in \overline{B}_{\mathcal{H}}(R)$ it holds $d\mathbb{P} \times dt$-a.e.,

$$\begin{aligned}
|\Gamma(Y)_t - \Gamma(Y'_t)| &\leq \mathbb{E}\left[\int_t^T |f(s,Y_s) - f(s,Y'_s)|\,ds \,\middle|\, \mathcal{F}_t\right] \\
&\leq L\mathbb{E}\left[\int_t^T |Y_s - Y'_s|\,ds \,\middle|\, \mathcal{F}_t\right] \\
&\leq L(T-t)\|Y - Y'\|_{L^\infty} \\
&\leq L\delta\|Y - Y'\|_{L^\infty} = \frac{1}{2}\|Y - Y'\|_{L^\infty}.
\end{aligned}$$

Furthermore, Γ maps $\overline{B}_{\mathcal{H}}(R)$ into itself: For all $Y \in \overline{B}_{\mathcal{H}}(R)$ it holds $d\mathbb{P} \times dt$-a.e.,

$$\begin{aligned}
|\Gamma(Y)_t| &\leq |\Gamma(Y)_t - \Gamma(0)_t| + |\Gamma(0)_t| \\
&\leq \|\Gamma(Y) - \Gamma(0)\|_{L^\infty} + \|\eta\|_{L^\infty} + (T-t)\|f(\cdot,0)\|_{L^\infty} \\
&\leq \frac{1}{2}\|Y\|_{L^\infty} + \|\eta\|_{L^\infty} + T\|f(\cdot,0)\|_{L^\infty} \leq R.
\end{aligned}$$

Hence, Γ has a unique fixed point in $\overline{B}_{\mathcal{H}}(R)$. By the martingale representation theorem, this fixed point gives the desired solution. $\square$

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 20. Dezember 2016 Paulwin Graewe